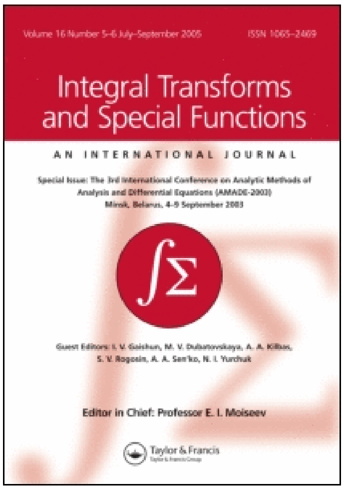

Integral Transforms and Special Functions
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713643686

Effective solution of a linear system with Chebyshev coefficients

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First Published:August2009

To cite this Article Kujan, Petr, Hromík, Martin and Šebek, Michael(2009)'Effective solution of a linear system with Chebyshev coefficients',Integral Transforms and Special Functions,20:8,619 — 628

To link to this Article: DOI: 10.1080/10652460902727938

URL: http://dx.doi.org/10.1080/10652460902727938

Integral Transforms and Special Functions
Vol. 20, No. 8, August 2009, 619–628

Effective solution of a linear system with Chebyshev coefficients

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(*Received 24 November 2008; final version received 5 January 2009*)

This paper presents an efficient algorithm for a special triangular linear system with Chebyshev coefficients. We present two methods of derivations, the first is based on formulae where the nth power of x is solved as the sum of Chebyshev polynomials and modified for a linear system. The second deduction is more complex and is based on the Gauss–Banachiewicz decomposition for orthogonal polynomials and the theory of hypergeometric functions which are well known in the context of orthogonal polynomials. The proposed procedure involves $\mathcal{O}(nm)$ operations only, where n is matrix size of the triangular linear system L and m is number of the nonzero elements of vector b. Memory requirements are $\mathcal{O}(m)$, and no recursion formula is needed. The linear system is closely related to the optimal pulse-wide modulation problem.

Keywords: orthogonal Chebyshev polynomials; hypergeometric functions; linear system; optimal PWM problem

1. Problem statement

Develop an algorithm with the complexity of $\mathcal{O}(nm)$ operations for the special triangular linear system $Lx = b$:

$$L = \begin{bmatrix} t_{0,0} & 0 & 0 & 0 & \dots & 0 \\ 0 & t_{1,1} & 0 & 0 & \dots & 0 \\ t_{2,0} & 0 & t_{2,2} & 0 & \dots & 0 \\ 0 & t_{3,1} & 0 & t_{3,3} & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & \dots & t_{n,n} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_0 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad (1)$$

where $t_{i,j}$ is the jth coefficient of the Chebyshev polynomial of degree i and b_i is an arbitrary real number. Note that the standard recursive algorithm involves $\mathcal{O}(n^2)$ operations.

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ISSN 1065-2469 print/ISSN 1476-8291 online

DOI: 10.1080/10652460902727938
http://www.informaworld.com

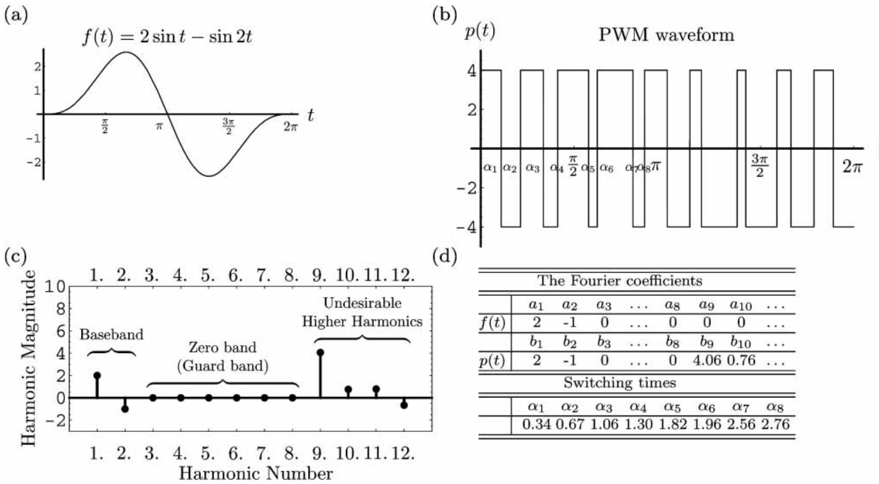

Figure 1. Solution for illustrative example – odd optimal PWM problem. (a) Example of odd symmetry 2π periodic function $f(t) = 2\sin t - \sin 2t$. (b) Optimal odd PWM waveform $p(t)$ for generation of odd symmetry periodic signal $f(t)$. (c) Amplitude spectrum of $p(t)$. (d) The solution for $p(t)$ and $f(t)$.

2. Motivation

The problem arises when looking for the optimal pulse-wide modulation (PWM) switching sequence $\alpha_1, \ldots, \alpha_n$ such that the output PWM waveform $p(t)$ has a required frequency spectrum, see [4,5,12]. The baseband frequency of $p(t)$ is equal to the frequency spectrum of a given $f(t)$, and a certain number of the following higher harmonics are equal to zero.

In the illustrative example, the amplitude frequency spectrum of signal $f(t) = 2\sin t - \sin 2t$ (Figure 1a) is $\{a_1 = 2, a_2 = -1\}$ (i.e., the first and second harmonics; the following harmonics are zero). The baseband frequency spectrum for $p(t)$ (Figure 1b) is chosen in the same way as the frequency spectrum $f(t)$ and then the following six harmonics (from the third to the eighth) are put equal to zero (Figure 1c). Thus, it is required that $\{b_1 = 2, b_2 = -1, b_3 = \ldots = b_8 = 0\}$ holds for the first eight harmonics of $p(t)$. The other higher harmonics $p(t)$ that are impossible to change by computing of switching angles $\alpha_1, \ldots, \alpha_8$ are $\{b_9 = 4.06, b_{10} = 0.76, b_{11} = 0.79, b_{12} = -0.66, \ldots\}$. The complete solution with numerical values is shown in Figure 1d.

The undesirable higher harmonics beyond the zero band are to be cancelled by a suitable filter. Obviously, the wider the band of zero harmonics, the better the result of filtration will be achieved. This demand calls for large systems of the type (1). In addition, for online implementation (e.g., in active filters, see [5,10]) fast calculation is required. A dedicated efficient algorithm for the set (1), fully exploiting its structure, is therefore needed.

3. Preliminaries

3.1. *Chebyshev polynomials of the first kind*

DEFINITION 3.1 (Chebyshev polynomial of the first kind [1, Chapter 22; 9]) *The orthogonal* Chebyshev polynomials *of the first kind $T_n(x)$ are defined by the trigonometric identity*

$$T_n(x) = \cos(n \arccos x) \quad \textit{whence} \quad T_n(\cos\theta) = \cos(n\theta) \tag{2}$$

or, alternatively, by the three-term recurrence relation

$$T_0(x) = 1,\, T_1(x) = x,\, T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Remark 3.2 The first few Chebyshev polynomials of the first kind are

$$\begin{aligned} &T_0(x) = 1, \quad T_2(x) = -1 + 2x^2, \quad T_4(x) = 1 - 8x^2 + 8x^4,\\ &T_1(x) = x, \quad T_3(x) = -3x + 4x^3, \quad T_5(x) = 5x - 20x^3 + 16x^5. \end{aligned}$$

COROLLARY 3.3 ([1, Chapter 22]) *The Chebyshev polynomials $T_n(x)$ satisfy various properties and identities.*

- *The Chebyshev polynomials $T_n(x)$ defined in terms of the sums*

$$T_n(x) = \frac{n}{2}\sum_{r=0}^{\lfloor n/2\rfloor} \frac{(-1)^r}{n-r}\binom{n-r}{r}(2x)^{n-2r}. \tag{3}$$

Therefore, the ith coefficient $t_{n,i}$ of n degree Chebyshev polynomial $T_n(x) = \sum_{i=0}^{n} t_{n,i}x^i$ is

$$\text{odd degree } n: \quad t_{n,i} = \begin{cases} 0 & \text{for } i = 0, 2, 4, \ldots, n-1,\\ K_{n,i} & \text{for } i = 1, 3, 5, \ldots, n, \end{cases} \tag{4a}$$

$$\text{even degree } n: \quad t_{n,i} = \begin{cases} K_{n,i} & \text{for } i = 0, 2, 4, \ldots, n,\\ 0 & \text{for } i = 1, 3, 5, \ldots, n-1, \end{cases} \tag{4b}$$

where

$$K_{n,i} = 2^i(-1)^{(n-i)/2}\frac{n}{n+i}\binom{(n+i)/2}{(n-i)/2}. \tag{4c}$$

Thus, we can rewrite the previous formulae to the form

$$\begin{aligned} t_{2n-1,2i-1} &= 2^{2i-1}(-1)^{n-i}\frac{2n-1}{2(n+i-1)}\binom{n+i-1}{n-i},\\ t_{2n,2i} &= 2^{2i}(-1)^{n-i}\frac{n}{n+i}\binom{n+i}{n-i}, \end{aligned} \tag{5a}$$

$$\begin{aligned} t_{2n-1,2i-2} = t_{2n,2i-1} &= 0, \quad n = 1, 2, \ldots, \quad i = 1, 2, \ldots, n,\\ t_{0,0} &= 1. \end{aligned} \tag{5b}$$

- *The orthogonality condition with respect to the weighting function $w(x) = 1/\sqrt{1-x^2}$ is*

$$\int_{-1}^{1} T_m(x)T_n(x)w(x)\mathrm{d}x = \begin{cases} \dfrac{1}{2}\pi\,\delta_{nm}, & m \neq 0, n \neq 0,\\ \pi, & m = n = 0, \end{cases} \tag{6}$$

where δ_{nm} is the Kronecker delta.

- *The moments are*

$$\mu_i = \mathcal{L}[x^i] = \int_{-1}^{1} x^i w(x)\mathrm{d}x = \begin{cases} \sqrt{\pi}\dfrac{\Gamma(i/2+1/2)}{\Gamma(i/2+1)}, & i = 0, 2, 4, \ldots, \\ 0, & i = 1, 3, 5, \ldots, \end{cases} \tag{7}$$

where $\Gamma(z)$ *is Gamma function* (*see* [*p.* 255–258] *or* [2]) *and* $\mathcal{L}[\cdot]$ *is a linear functional* (*see* [3, *Chapter* 2] *or* [8, *Chapter* 7]). *Thus,*

$$\mu_{2i} = 2^{-i}\pi\frac{(2i-1)!!}{i!} = 2^{-2i+1}\pi\binom{2i-1}{i-1}, \tag{8a}$$

$$\mu_{2i+1} = 0, \quad i = 0, 1, 2, \ldots. \tag{8b}$$

Note that $\mathcal{L}[T_m(x)T_n(x)] = \int_{-1}^{1} T_m(x)T_n(x)w(x)\mathrm{d}x$.

PROPERTY 3.4 (Powers of x in terms of $T_n(x)$ [9, Chapter 2.3.1]) *The power* x^n *can be expressed in terms of the Chebyshev polynomials of degree up to* n *by the following formulas:*

$$\textit{odd degree } n: \quad x^n = 2^{1-n}\sum_{k=0}^{(n-1)/2}\binom{n}{k}T_{n-2k}(x), \tag{9a}$$

$$\textit{even degree } n: \quad x^n = 2^{-n}\binom{n}{n/2}T_0(x) + 2^{1-n}\sum_{k=0}^{n/2-1}\binom{n}{k}T_{n-2k}(x). \tag{9b}$$

Proof See [9, Chapter 2.3.1]. ■

3.2. *Gauss–Banachiewicz decomposition*

THEOREM 3.5 (Gauss–Banachiewicz decomposition [3, p. 78]) *The moments Hankel matrix*

$$H = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & & \iddots & \vdots \\ \vdots & \iddots & & \mu_{2k-1} \\ \mu_k & \cdots & \mu_{2k-1} & \mu_{2k} \end{bmatrix}$$

can be factored as $LHL^{\mathrm{T}} = D$, *where for the orthogonal polynomial* $T_k(x) = t_{k,k}x^k + t_{k,k-1}x^{k-1} + \ldots + t_{k,0}$ *is*

$$L = \begin{bmatrix} t_{0,0} & & & 0 \\ t_{1,0} & t_{1,1} & & \\ \vdots & \vdots & \ddots & \\ t_{k,0} & t_{k,1} & \ldots & t_{k,k} \end{bmatrix}$$

and D *is diagonal matrix* $D = \mathrm{diag}(\mathcal{L}[T_0^2(x)], \mathcal{L}[T_1^2(x)], \mathcal{L}[T_2^2(x)], \ldots, \mathcal{L}[T_k^2(x)])$.

3.3. *Special functions*

DEFINITION 3.6 (Hypergeometric functions [6]) *A generalized hypergeometric function*

$$ {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| x\right] $$

has a series representation

$$ \sum_{k=0}^{\infty} c_k $$

with c_{k+1}/c_k a rational function of k. The ratio c_{k+1}/c_k can be factored, and is usually written as

$$ \frac{c_{k+1}}{c_k} = \frac{(k+a_1)\cdots(k+a_p)x}{(k+b_1)\cdots(k+b_q)(k+1)}. $$

For $c_0 = 1$, the previous equation can be solved for c_n as

$$ c_n = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!} $$

and

$$ {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| x\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}, $$

where $(x)_n$ is the Pochhammer symbol.

DEFINITION 3.7 (Pochhammer symbol [7]) *The Pochhammer symbol (or shifted or rising or upper factorial) is*

$$ (x)_n = x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \textit{for} \quad n \geq 0. $$

PROPERTY 3.8 *If x is a negative integer $-m$, then the Pochhammer symbol satisfies*

$$ (x)_n = (-m)_n = (-1)^n (m-n+1)_n \qquad \textit{for} \quad m \geq n, \tag{10a} $$

$$ (x)_n = 0 \qquad \textit{for} \quad m < n. \tag{10b} $$

THEOREM 3.9 (Saalschütz's theorem [11])

$$ {}_3F_2\left[\begin{matrix} a, b, -n \\ c, 1+a+b-c-n \end{matrix}\middle| 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}, $$

where n is a positive integer.

4. Solving the linear system with Chebyshev coefficients

PROPOSITION 4.1 [Linear system with Chebyshev coefficients] *Let us have a linear system*

$$Lx = b, \tag{11}$$

where L is a lower triangular matrix with Chebyshev coefficients and the right-hand side is a vector b in the form

$$L = \begin{bmatrix} t_{0,0} & 0 & 0 & 0 & \dots & 0 \\ 0 & t_{1,1} & 0 & 0 & \dots & 0 \\ t_{2,0} & 0 & t_{2,2} & 0 & \dots & 0 \\ 0 & t_{3,1} & 0 & t_{3,3} & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & \dots & t_{n,n} \end{bmatrix} \quad and \quad b = \begin{bmatrix} b_0 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$t_{k,i}$ is the ith coefficient of Equation (4) *of the Chebyshev polynomial $T_k(x)$ of degree k and b_i, $i = 0, 1, \dots, m (m \le n)$ is an arbitrary real number and $b_i = 0$, $i = m+1, \dots, n$. Then the solution $x = (x_0, x_1, x_2, \dots, x_n)^{\mathrm{T}}$ of this system is*

$$x_{2i} = 2^{-2i}\binom{2i}{i}b_0 + 2^{1-2i}\sum_{k=1}^{\min\{i,\lfloor m/2\rfloor\}}\binom{2i}{i-k}b_{2k}, \quad i = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor, \tag{12a}$$

$$x_{2i-1} = 2^{2-2i}\sum_{k=1}^{\min\{i,\lceil m/2\rceil\}}\binom{2i-1}{i-k}b_{2k-1}, \quad i = 1, \dots, \left\lceil \frac{n}{2} \right\rceil. \tag{12b}$$

We provide two proofs: the first one is a direct application of the Property 3.4, it is intuitive and easy to understand; the second one is more complex but it gives a connection between Chebyshev orthogonal polynomials, Gauss–Banachiewicz decomposition and hypergeometric functions, for more details see [8].

Proof of Proposition 4.1 (1) First, for odd n according to Equation (9a) we have

$$x_n = 2^{1-n}\sum_{k=0}^{(n-1)/2}\binom{n}{k}T_{n-2k}(\overline{x}),$$

where we replaced the ith power of x in Chebyshev polynomials by variable x_i, thus $T_{n-2k}(\overline{x}) = \sum_{j=0}^{n-2k} t_{n-2k,j}x_j$. But the $T_{n-2k}(\overline{x})$ is the $(n-2k)$th row in matrix L in Equation (11) and is equal to b_{n-2k}, thus we can directly solve the unknown variable x_n as $x_n = 2^{1-n}\sum_{k=0}^{(n-1)/2}\binom{n}{k}b_{n-2k}$. According to the assumption that n is an odd integer, we can rewrite this solution as the following expression $x_{2i-1} = 2^{-2i+2}\sum_{k=0}^{i-1}\binom{2i-1}{k}b_{2(i-k)-1}$ and if we take the fact that for $n-2k > m$ is $b_{n-2k} = 0$ and re-index k in the sum

$$\begin{aligned} 2^{2i-2}x_{2i-1} &= \sum_{k=0}^{i-1}\binom{2i-1}{-(i-k)+i}b_{2(i-k)-1} = |i-k=m| \\ &= \sum_{i-m=0}^{i-m=i-1}\binom{2i-1}{i-m}b_{2m-1} = \sum_{m=1}^{i}\binom{2i-1}{i-m}b_{2m-1}, \end{aligned}$$

we exactly obtained Equation (12b). Similarly, according to Equation (9b) we obtain the solution for even n.

(2) As the matrix L satisfies the conditions of the Theorem 3.5, we introduce a new variable y and put $x = HL^{\mathrm{T}}y$. Then $Lx = LHL^{\mathrm{T}}y = Dy = b$ recall that

$$H = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & & \iddots & \vdots \\ \vdots & \iddots & & \mu_{2n-1} \\ \mu_n & \cdots & \mu_{2n-1} & \mu_{2n} \end{bmatrix},$$

where μ_i's are given by Equation (8) and according to Equation (6)

$$D = \mathrm{diag}(d_0, d_1, \ldots, d_n), \quad \text{where } d_0 = \pi \text{ and } d_i = \frac{\pi}{2}, \quad i = 1, \ldots, n.$$

The solution of the diagonal system $Dy = b$ is straightforward, $y = (2/\pi)((1/2)b_0, b_1, \ldots, b_m, 0, \ldots, 0)$. Now we compute the product HL^{T} and denote it as the matrix C of the size $n+1 \times n+1$

$$C = (c)_{ij}$$
$$= \begin{pmatrix} \mu_0 t_{0,0} & 0 & \mu_0 t_{2,0} + \mu_2 t_{2,2} & 0 & \mu_0 t_{4,0} + \mu_2 t_{4,2} + \mu_4 t_{4,4} & \cdots \\ 0 & \mu_2 t_{1,1} & 0 & \mu_2 t_{3,1} + \mu_4 t_{3,3} & 0 & \cdots \\ \mu_2 t_{0,0} & 0 & \mu_2 t_{2,0} + \mu_4 t_{2,2} & 0 & \mu_2 t_{4,0} + \mu_4 t_{4,2} + \mu_6 t_{4,4} & \cdots \\ 0 & \mu_4 t_{1,1} & 0 & \mu_4 t_{3,1} + \mu_6 t_{3,3} & 0 & \cdots \\ \mu_4 t_{0,0} & 0 & \mu_4 t_{2,0} + \mu_6 t_{2,2} & 0 & \mu_4 t_{4,0} + \mu_6 t_{4,2} + \mu_8 t_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Generally, the $(c)_{ij}$ element of the matrix C

$$c_{2i-1,2j-1} = \sum_{k=0}^{j-1} \mu_{2(k+i-1)} t_{2(j-1),2k}, \quad i, j = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil, \tag{13a}$$

$$c_{2i,2j} = \sum_{k=1}^{j} \mu_{2(k+i-1)} t_{2j-1,2k-1}, \quad i, j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, \tag{13b}$$

$$c_{2i,2j-1} = 0, \quad i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, \quad j = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil, \tag{13c}$$

$$c_{2i-1,2j} = 0, \quad i = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil, \quad j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\lfloor \tag{13d}$$

and after substitution of Equations (5) and (8) it follows that

$$c_{2i-1,2j-1} = (-1)^{j-1} 2^{-2i+3} \pi (j-1) \mathcal{S}'_{ij}, \quad i, j = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil,$$
$$\text{with } \mathcal{S}'_{ij} = \sum_{k=0}^{j-1} (-1)^k \frac{1}{j+k-1} \binom{2(k+i)-3}{k+i-2} \binom{j+k-1}{j-k-1}, \tag{14a}$$

$$c_{2i,2j} = (-1)^j 2^{-2i+2} \pi (2j-1) \mathcal{S}''_{ij}, \quad i, j = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,$$
$$\text{with } \mathcal{S}''_{ij} = \sum_{k=1}^{j} (-1)^k \frac{1}{2(j+k-1)} \binom{2(k+i)-3}{k+i-2} \binom{j+k-1}{j-k}. \tag{14b}$$

The sums $\mathcal{S}'_{ij}, \mathcal{S}''_{ij}$ in the previous expressions are binomial sums and according to the Definition 3.6, we can rewrite it as the hypergeometric function ${}_3F_2$:

$$\mathcal{S}_1 = \frac{2^{2i-3}}{(j-1)\sqrt{\pi}}\frac{\Gamma(i-1/2)}{\Gamma(i)}{}_3F_2\left[\left.\begin{matrix} i-\frac{1}{2}, j-1, -(j-1) \\ \frac{1}{2}, i \end{matrix}\right| 1\right], \tag{15a}$$

$$\mathcal{S}_2 = -\frac{2^{2i-2}}{\sqrt{\pi}}\frac{\Gamma(i+1/2)}{\Gamma(i+1)}{}_3F_2\left[\left.\begin{matrix} i+\frac{1}{2}, j, -(j-1) \\ \frac{3}{2}, i+1 \end{matrix}\right| 1\right]. \tag{15b}$$

The obtained hypergeometric functions correspond to the conditions of the the Theorem 3.9, therefore using Equation (10) we have

$$\begin{aligned}\mathcal{S}_1 &= \frac{2^{2i-3}}{(j-1)\sqrt{\pi}}\frac{\Gamma(i-1/2)}{\Gamma(i)}\frac{(-(i-1))_{j-1}(3/2-j)_{j-1}}{(1/2)_{j-1}(-(i+j-2))_{j-1}} \\ &= \frac{2^{2i-3}}{(j-1)\sqrt{\pi}}\frac{\Gamma(i-1/2)}{\Gamma(i)}\frac{(i-j+1)_{j-1}(3/2-j)_{j-1}}{(1/2)_{j-1}(i)_{j-1})}, \end{aligned} \tag{16a}$$

$$\begin{aligned}\mathcal{S}_2 &= -\frac{2^{2i-2}}{\sqrt{\pi}}\frac{\Gamma(i+1/2)}{\Gamma(i+1)}\frac{(-(i-1))_{j-1}(-(j-3/2))_{j-1}}{(3/2)_{j-1}(-(i+j-1))_{j-1}} \\ &= -\frac{2^{2i-2}}{\sqrt{\pi}}\frac{\Gamma(i+1/2)}{\Gamma(i+1)}\frac{(-1)^{j-1}(i-j+1)_{j-1}(1/2)_{j-1}}{(3/2)_{j-1}(i+1)_{j-1}}, \end{aligned} \tag{16b}$$

and consequently

$$\begin{aligned}c_{2i-1,2j-1} &= 2^{2-2i}\pi\frac{\Gamma(2i-1)}{\Gamma(i-j+1)\Gamma(i+j-1)} = 2^{2-2i}\pi\frac{(2i-2)!}{(i-j)!(i+j-2)!} \\ &= 2^{2-2i}\pi\binom{2i-2}{i-j}, \quad i,j = 1,2,\ldots,\left\lceil\frac{n}{2}\right\rceil, \end{aligned} \tag{17a}$$

$$\begin{aligned}c_{2i,2j} &= 2^{1-2i}\pi\frac{\Gamma(2i)}{\Gamma(i-j+1)\Gamma(i+j)} = 2^{1-2i}\pi\frac{(2i-1)!}{(i-j)!(i+j-1)!} \\ &= 2^{1-2i}\pi\binom{2i-1}{i-j}, \quad i,j = 1,2,\ldots,\left\lfloor\frac{n}{2}\right\rfloor. \end{aligned} \tag{17b}$$

As the solution (17) contains binomial coefficient $\binom{\cdots}{i-j}$, $c_{2i-1,2j-1} = c_{2i,2j} = 0$ for all $j > i$. Finally, we obtain

$$x = Cy = \frac{2}{\pi}\begin{bmatrix} c_{1,1} & 0 & 0 & 0 & \ldots & 0 \\ 0 & c_{2,2} & 0 & 0 & \ldots & 0 \\ c_{3,1} & 0 & c_{3,3} & 0 & \ldots & 0 \\ 0 & c_{4,2} & 0 & c_{4,4} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & c_{n,2} & 0 & c_{n,4} & & c_{n,n} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2}b_0 \\ b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \end{bmatrix}$$

and further

$$x_{2i} = \frac{1}{\pi} c_{2i+1,1} b_0 + \frac{2}{\pi} \sum_{k=1}^{\min\{i,\lfloor m/2 \rfloor\}} c_{2i+1,2k+1} b_{2k}, \quad i = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,$$

$$x_{2i-1} = \frac{2}{\pi} \sum_{k=1}^{\min\{i,\lceil m/2 \rceil\}} c_{2i,2k} b_{2k-1}, \quad i = 1, \ldots, \left\lceil \frac{n}{2} \right\rceil.$$

After the substitution of Equation (17) we arrive at expressions (12). ■

Remark 4.2 The result of Equation (12) consists of two independent solutions (this is also clear directly from the matrix L in Equation (11)). The first solution of Equation (12a) corresponds to even coefficients of x given by the linear system (matrices in this form are written for odd m and odd n)

$$\begin{bmatrix} t_{0,0} & 0 & 0 & \ldots & 0 & 0 \\ t_{2,0} & t_{2,2} & 0 & \ldots & 0 & 0 \\ t_{4,0} & t_{4,2} & t_{4,4} & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-3,0} & t_{n-3,2} & t_{n-3,4} & \ldots & t_{n-3,n-3} & 0 \\ t_{n-1,0} & t_{n-1,2} & t_{n-1,4} & \ldots & t_{n-3,n-3} & t_{n-1,n-1} \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ \vdots \\ x_{n-3} \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_2 \\ \vdots \\ b_{m-1} \\ 0 \\ \vdots \end{bmatrix}.$$

The second solution of Equation (12b) correspond to odd coefficients of x

$$\begin{bmatrix} t_{1,1} & 0 & 0 & \ldots & 0 & 0 \\ t_{3,1} & t_{3,3} & 0 & \ldots & 0 & 0 \\ t_{5,1} & t_{5,3} & t_{5,5} & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-2,1} & t_{n-2,3} & t_{n-2,5} & \ldots & t_{n-2,n-2} & 0 \\ t_{n,1} & t_{n,3} & t_{n,5} & \ldots & t_{n,n-2} & t_{n,n} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \\ x_{n-2} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_3 \\ \vdots \\ b_m \\ 0 \\ \vdots \end{bmatrix}. \tag{18}$$

Remark 4.3 Special case for $m = 1$. We obtain the solution directly from Equation (12) and it reads

$$x_{2i} = 2^{-2i} \binom{2i}{i} b_0, \quad i = 0, 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,$$

$$x_{2i-1} = 2^{2(1-i)} \binom{2i-1}{i-1} b_1, \quad i = 1, 2, 3, \ldots, \left\lceil \frac{n}{2} \right\rceil.$$

The last expression is the ‘sum of powers’ for optimal PWM problem for single phase inverter; see [5].

5. Conclusion and algorithm complexity

The presented algorithm is nonrecursive and requires only $\mathcal{O}(nm)$ operations (additions and multiplications) compared with the standard recursive procedure for general triangular system,

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} t_{i,j} x_j}{t_{i,i}}, \quad i = 1, 2, \ldots, n,$$

which involves $\mathcal{O}(n^2)$ operations. Memory requirements are determined only by the vector b, and only $\mathcal{O}(m)$ memory space is therefore required. Generating and storing of Chebyshev coefficients takes no extra memory.

Acknowledgements

The work was supported by the Ministry of Education of the Czech Republic, Project 1M0567.

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