Conditional Belief Functions: a Comparison among Different Definitions

Giulianella Coletti
Marcello Mastroleo
Dipartimento di Matematica e Informatica
University of Perugia
(coletti,mastroleo)@dipmat.unipg.it

Abstract

By following the idea that a conditional measure is more than a sort of a rearrangement of an unconditional measure, we are interested in studying the controversial concept of conditional belief function as a *generalized decomposable conditional measure*, that is a suitable real function $\varphi(\cdot|\cdot)$, defined on a family of conditional events, ruled by a set of axioms involved two composition rules $\oplus$ and $\odot$.

1 Introduction

Starting from the pioneering papers by de Finetti ([11], [12]), Popper [16], Rényi [17] (for probability), Cox [10] (for general measures), many authors discussed about the convenience to give a direct definition of a conditional measure (for a thorough exposition and bibliography, see Halpern [14]). In the last years, starting from the concept of *conditional event* $E|H$, represented by a suitable *three–valued random variable* whose values are $1, 0, t(E|H)$, the latter turning out to be the relevant conditional measure $\varphi(\cdot|\cdot)$, a new methodology to discover axioms for defining conditional measures has been presented and by it a general definition of decomposable conditional uncertainty measure has been introduced (see for instance [4] or [6]).

Then different (decomposable) conditional measures can be obtained by particular choices of the two operations $\oplus$ and $\odot$. For example, choosing ordinary sum and product ([3]), *max* and *min* ([1]), or *max* and any t-norm T ([2]), we get, respectively, finitely additive conditional probability, conditional possibility or generalized conditional possibility. The definitions introduced by following this process generalize the partial ones, obtained by means the restriction of an unconditional measure to a conditioning event. Moreover the systems of axioms are a tool for discovering and removing the inconsistencies produced by introducing ad hoc rules to cover the lacks due to the partial definitions and so to extend it to all the pairs of events.

In [4], in particular, the interest has been focused on searching the minimal (necessary and sufficient) conditions for $\oplus$ and $\odot$ which render a conditional measure $\varphi(\cdot|\cdot)$ formally (or, better, essentially) "similar" to a conditional probability, in the sense that it can be represented in terms of *classes* of $\oplus$-decomposable uncertainty measures.

By using the above framework, at WUPES 2004 Conference a definition of conditional belief has been presented by Coletti and Scozzafava ([7]).

Nevertheless it generalizes only one of the many definitions present in the literature, obtained starting from an unconditional belief function (see for instance [13], [15], [18], [19])

The main aim of this paper is to study the different definitions, or better their possible generalizations (if exist), as $(\oplus, \odot)$-decomposable measures and to analyze, in this context, their properties in terms of properties of the operation $(\oplus, \odot)$.

2 Preliminaries

In this section we briefly recall the framework of reference, the connection between coherent conditional probabilities and belief function and finally some of the (partial) conditioning rules known in the literature.

2.1 From Conditional Events to Conditional Measures

In order to deal adequately with conditional *measures*, we need to introduce the concept of *conditional* event, denoted by $E|H$, with $H \neq \emptyset$ (where $\emptyset$ is the *impossible* event): it is a *generalization* of the concept of event, and can be defined through its *truth–value* $T(E|H)$. *When we assume* H *true*, we take $T(E|H)$ equal to 1 or 0 according to whether E or its contrary E^c is true, and *when we assume that* H *is false* we take $T(E|H)$ equal to a suitable function $t(E|H)$ with values in $[0, 1]$.

This truth–value $T(E|H)$ extends the concept of *indicator* $I_E = T(E|\Omega)$ concerning an (unconditional) event E.

Since the conditional event $E|H$, or better its *boolean support* that is the (ordered) pair (E, H), induces a (unique) partition of the certain event Ω, that is $(E \wedge H, E^c \wedge H, H^c)$: one puts in fact

$$t(E|H) = t(E \wedge H, E^c \wedge H, H^c)\,.$$

It follows $t(E|H) = t(E \wedge H|H)$, and so $T(E|H) = T\big((E \wedge H)|H\big)$.

In conclusion we require for $t(.|.)$ only the following conditions:

i) the function $t(E|H)$ depends only on the partition $E \wedge H, E^c \wedge H, H^c$.

ii) the function $t(\cdot|H)$ must be *not identically equal to zero*).

A useful representation of $T(E|H)$ (that will be denoted from now on by $I_{E|H}$) can be given by means of a discrete real random quantity

$$(*) \qquad I_{E|H} = 1 \cdot I_{E\wedge H} + 0 \cdot I_{E^c \wedge H} + t(E|H) \cdot I_{H^c}\,.$$

In [4] has been showed that, by introducing suitable (partial) operations among *conditional events*, the choice of these operations determines the various *conditional measures* $t(E|H)$ representing uncertainty. More precisely, if we operate only with those elements of $\mathsf{T}\times\mathsf{T}$ such that the range of each operation is T, we get "automatically" (so to say), conditions on $t(E|H)$ that can be regarded as the "natural" axioms for a conditional measure φ defined on $\mathcal{C} = \mathcal{E} \times \mathcal{H}^o$, with $\mathcal{H}^o = \mathcal{H} \setminus \{\emptyset\}$, i.e.

(C1) $\varphi(E|H) = \varphi(E \wedge H|H)$, for every $E \in \mathcal{E}$ and $H \in \mathcal{H}^o$,

(C2) for any given $H \in \mathcal{H}^o$ and for any $E,\, A \in \mathcal{E}$, with $A \wedge E \wedge H \neq \emptyset$, we have

$$\varphi\big((E \vee A)|H\big) = \varphi(E|H) \oplus \varphi(A|H)\ ,\ \varphi(\Omega|H) = 1\ ,\ \varphi(\emptyset|H) = 0\,,$$

(C3) for every $A \in \mathcal{E}$ and $E,\, H,\, E \wedge H \in \mathcal{H}^o$,

$$\varphi\big((E \wedge A)|H\big) = \varphi(E|H) \odot \varphi\big(A|(E \wedge H)\big)\,.$$

Then different (decomposable) conditional measures can be obtained by particular choices of the two operations $\oplus$ and $\odot$. For example, choosing ordinary sum and product (see[3]), or *max* and *min* (see[?]), or *max* and a t-norm T (see[2]), we get, respectively, conditional probability, conditional possibility, or generalized conditional possibility.

The above properties hold also for *arbitrary* sets $\mathcal{E}$ and $\mathcal{H}$, but a sensible definition of conditional measure requires to put "natural" algebraic structures on these sets.

Definition 1 *Let φ be a real function defined on $\mathcal{C} = \mathcal{E} \times \mathcal{H}^o$, with $\mathcal{E}$ a Boolean algebra, $\mathcal{H} \subseteq \mathcal{E}$ an additive set (i.e., closed with respect to finite logical sums) and $\mathcal{H}^o = \mathcal{H} \setminus \{\emptyset\}$, and denote by $\varphi(\mathcal{C})$ the range of φ. Then the function φ is a $(\oplus, \odot)$–decomposable* conditional *measure if there exist two commutative, associative and increasing operations $\oplus, \odot$ from $\varphi(\mathcal{C}) \times \varphi(\mathcal{C})$ to $\mathbb{R}^+$, having, respectively,* 0 *and* 1 *as neutral elements, and with $\odot$ distributive over $\oplus$, such that (C1), (C2), (C3) hold.*

To deal with more general conditional measures (such as conditional beliefs) we need to introduce the definition of *generalized* $(\oplus, \odot)$*–decomposable conditional measure*(given in [7]), which extends that of $(\oplus, \odot)$*–decomposable conditional measure*.

Definition 2 *Given a family $\mathcal{C} = \mathcal{E} \times \mathcal{H}^0$ of conditional events, where $\mathcal{E}$ is a Boolean algebra, $\mathcal{H}$ an additive set, with $\mathcal{H} \subseteq \mathcal{E}$ and $\mathcal{H}^0 = \mathcal{H}\setminus\{\emptyset\}$, a real function φ defined on $\mathcal{C}$ is a* generalized $(\oplus, \odot)$*–decomposable* conditional *measure if*

(γ_1) $\varphi(E|H) = \varphi(E \wedge H|H)$, for every $E \in \mathcal{E}$ and $H \in \mathcal{H}^o$,

(γ_2) for any given $H \in \mathcal{H}^o$, it is $\varphi(\Omega|H) = 1$, $\varphi(\emptyset|H) = 0$ and $\varphi(\cdot|H)$ is a capacity*; there exists an operation $\oplus : \{\varphi(\mathcal{C})\}^2 \to \varphi(\mathcal{C})$ whose restriction to the set*

$$\Delta = \{(\varphi(E_i|H), \varphi(E_j|H)) : E_i, E_j \in \mathcal{E},\, H \in \mathcal{H}^0,\, E_i \wedge E_j \wedge H = \emptyset\}$$

is such that

$$\varphi(E_i \vee E_j|H) = \varphi(E_i|H) \oplus \varphi(E_j|H)$$

for every $E_i, E_j \in \mathcal{E}$, with $E_i \wedge E_j = \emptyset$.

(γ_3) there exists an operation $\odot : \{\varphi(\mathcal{C})\}^2 \to \varphi(\mathcal{C})$ whose restriction to the set

$$\Gamma = \Big\{\Big(\varphi(E|H), \varphi\big(A|(E \wedge H)\big)\Big) : A \in \mathcal{E},\, E, H, E \wedge H \in \mathcal{H}^o\Big\}$$

is increasing, admits 1 as neutral element and is such that, for every $A \in \mathcal{E}$ and E, $H \in \mathcal{H}^o$, $E \wedge H \neq \emptyset$,

$$\varphi\big((E \wedge A)|H\big) = \varphi(E|H) \odot \varphi\big(A|(E \wedge H)\big)\,.$$

(γ_4) The operation $\odot$ is distributive over $\oplus$ only *for relations of the kind*

$$\varphi(H|K) \odot \Big(\varphi\big(E|(H \wedge K)\big) \oplus \varphi\big(F|(H \wedge K)\big)\Big)\,,$$

with $K, H \wedge K \in \mathcal{H}^0$, $E \wedge F \wedge H \wedge K = \emptyset$.

Remark 1 – *It is easily seen that, with respect to the elements of Δ and Γ, the operations $\oplus$ and $\odot$ are commutative and associative.*

We recall that in [7] it has been proved that there exists an operation $\oplus$ from $Bel(\mathcal{E}) \times Bel(\mathcal{E})$ to $Bel(\mathcal{E})$, with $\oplus$ increasing *only* with respect to pairs of events ordered by implication, that is in the set

$$\Big\{\Big(\varphi(A), \varphi(C)\Big), \Big(\varphi(B), \varphi(C)\Big) : A, B, C \in \mathcal{E},\, A \subset B,\, BC = \emptyset,\, Bel(B) > Bel(A)\Big\}\,,$$

such that Bel is a *generalized $\oplus$-decomposable measure* on $\mathcal{E}$. So, for the definition of conditional belief function as particular *generalized* $(\oplus, \odot)$–decomposable *conditional* measure, it is necessary only to focus on $\odot$ operation: in fact the first two axioms are

i) $Bel(E|H) = Bel(EH|H)$;

ii) $Bel(\cdot|H)$ is a belief function $\forall H \in \mathcal{K}$;

2.2 Lower coherent conditional probabilities and belief functions

We say that the assessment $P(\cdot|\cdot)$ on $\mathcal{C}$ is *coherent* if there exists $\mathcal{C}' \supset \mathcal{C}$, with $\mathcal{C}' = \mathcal{E} \times \mathcal{H}^o$ ($\mathcal{E}$ a Boolean algebra, $\mathcal{H}$ an additive set), such that $P(\cdot|\cdot)$ can be extended from $\mathcal{C}$ to $\mathcal{C}'$ as a *conditional probability*.

A fundamental result concerning coherent conditional probabilities is the following, essentially due (for unconditional events, i.e. the particular case in which all conditioning events coincide with Ω) to de Finetti[12].

Theorem 1 *Let $\mathcal{C}$ be a family of conditional events and P a corresponding assessment; then there exists a (possibly not unique) coherent extension of P to an* arbitrary *family $\mathcal{K} \supseteq \mathcal{C}$,* if and only if *$P$ is coherent on $\mathcal{C}$.*

Given an *arbitrary* set $\mathcal{C} = \mathcal{E} \times \mathcal{H}$ of conditional events (with $\emptyset \notin \mathcal{H}$), a *coherent lower conditional probability* on $\mathcal{C}$ is a nonnegative function $\underline{P}$ such that there exists a non-empty *dominating family* $\mathcal{P} = \{P(\cdot|\cdot)\}$ of *coherent* conditional probabilities on $\mathcal{C}$ whose lower envelope is $\underline{P}$, that is, for every $E|H \in \mathcal{C}$,

$$\underline{P}(E|H) = \inf_{\mathcal{P}} P(E|H) \,.$$

In particular, by taking $\mathcal{H} = \{\Omega\}$, we get a coherent lower probability on $\mathcal{E}$.

The next result, given in [7], shows that a coherent conditional lower probability $\underline{P}(\cdot\,|K)$ extending a coherent probability $P(H_i)$ – where the events H_i's are a partition of the certain event Ω and K is the union of some (possibly all) of them – is a belief function.

Theorem 2 *Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be a finite set of pairwise incompatible events. Denoting by $\mathcal{K}$ the additive set spanned by them, and given an algebra $\mathcal{A}$ of events, put $\mathcal{C} = \mathcal{A} \times \mathcal{K}$.*

If $P(\cdot)$ is a coherent probability on $\mathcal{H}$, let $\mathcal{P}$ be the class of coherent conditional probabilities $P(\cdot|\cdot)$ extending $P(\cdot)$ on $\mathcal{C}$. Consider, for $E|K \in \mathcal{C}$, the lower probability

$$\underline{P}(E|K) = \inf_{\mathcal{P}} P(E|K) \,; \tag{1}$$

then for any $K \in \mathcal{K}$ the function $\underline{P}(\cdot\,|K)$ is a belief function on $\mathcal{A}$.

It is possible to prove also a sort of viceversa of the previous theorem:

Theorem 3 – *Let $\mathcal{A}$ be a finite algebra of events and φ a belief function on $\mathcal{A}$ The there exists a partition $\mathcal{H} = \{H_1, \ldots, H_n\}$ of Ω and a (coherent) probability on $\mathcal{H}$ such that the lower envelope of the class of coherent conditional probabilities $P(\cdot|\cdot)$ extending $P(\cdot)$ on $\mathcal{C}$ coincides with φ on $\mathcal{A}$.*

The proof is essentially given in [7], section 3.1

2.3 Some conditioning rules

We recall now some conditional rules presents in the literature: obviously them are partial (from our point of view), and are given starting from an unconditional belief function.

Definition 3 *Let Bel be a function on an algebra of events $\mathcal{A}$, for any $B \in \mathcal{A}$ with $Bel(B) > 0$ and for every $A \in \mathcal{A}$ we have:*

$$Bel(A|B) = \frac{Bel(AB)}{Bel(B)} \tag{2}$$

Definition 4 (Dempster rule:) *Let Bel be a function on an algebra of events $\mathcal{A}$, for any $B \in \mathcal{A}$ such that $Pl(B) = 1 - Bel(B^c) > 0$ and for every $A \in \mathcal{A}$ we have:*

$$Bel(A|B) = \frac{Bel(A \vee B^c) - Bel(B^c)}{1 - Bel(B^c)} = \frac{Pl(B) - Pl(A^c B)}{Pl(B)}. \tag{3}$$

Definition 5 *Let Bel be a function on an algebra of events $\mathcal{A}$, for any $B \in \mathcal{A}$ such that $Pl(B) = 1 - Bel(B^c) > 0$ and for any $A \in \mathcal{A}$ we have:*

$$Bel(A|B) = \frac{Bel(A \vee B)}{Pl(B)}. \tag{4}$$

Definition 6 (Bayes rule:) *Let Bel be a function on an algebra of events $\mathcal{A}$, for all $B \in \mathcal{A}^*$ and for any $A \in \mathcal{A}$, such that $Bel(AB) + Pl(A^c B) > 0$ we have:*

$$Bel(A|B) = \frac{Bel(AB)}{Bel(AB) + Pl(A^c B)} \tag{5}$$

3 Conditional belief functions

We will discuss about the possibility to see each condition introduced in the previous section as a partial definition of a conditional belief function, element of the class of generalized $(\oplus, \odot)$–decomposable conditional measures with a particular choice of $\odot$.

First of all we note that the first two definition are strictly related, or better are in a sense "dual": in fact both them come from the choice of $\odot$ equal to the usual product, in a general definition of conditional belief or conditional plausibility, respectively, as $(\oplus, \odot)$–decomposable conditional measures.

So the Definitions 3 and 4 can become:

Definition 7 *A function Bel defined on $\mathcal{C} = \mathcal{E} \times \mathcal{H}^o$, with $\mathcal{H}^o = \mathcal{H} \setminus \{\emptyset\}$, is a conditional belief if satisfies condition (i), (ii) and the following:*

iii) For every $E \in \mathcal{A}$ and $H, K \in \mathcal{K}$

$$Bel(E|K) = Bel(E|H) \cdot Bel(H|K).$$

Definition 8 (Dempster conditional Belief) *A function Pl defined on $\mathcal{C} = \mathcal{E} \times \mathcal{H}^o$, with $\mathcal{H}^o = \mathcal{H} \setminus \{\emptyset\}$, is a conditional plausibility if satisfies the following conditions*

d) $Pl(E|H) = Pl(EH|H)$;

dd) $Pl(\cdot|H)$ *is a plausibility function* $\forall H \in \mathcal{K}$;

ddd) For every $E \in \mathcal{A}$ *and* $H, K \in \mathcal{K}$

$$Pl(EH|K) = Pl(E|HK) \cdot Pl(H|K).$$

Moreover, given a conditional Plausibility, a conditional belief function $Bel(\cdot|\cdot)$ *is defined by duality as follows: for every event* $E|H \in \mathcal{C}$

$$Bel(E|H) = 1 - Pl(E^c|H).$$

It is very easy to obtain by *ddd*), for $H = \Omega$ and $F = E^c$, the equality:

$$Pl(H)Bel(F|H) = Pl(H)Pl(F^cH)$$

and so condition (2) for H with $Pl(H) > 0$.

Obviously it is possible to give a "direct" axiomatization Dempster conditional Belief, in terms of conditional belief, but in this case the third axiom links conditional events different from the triple $EH|K, E|HK, H|K$. In fact an equivalent set of axioms to define Dempster conditional belief is $i), ii)$ and the following:

$iii)_D$ For every $E \in \mathcal{A}$ and $H, K \in \mathcal{K}$

$$Bel_D(E|K) = 1 - [1 - Bel_D(E|H)] \cdot [1 - Bel_D(H^c|K)]$$

Consider now condition in Definition 5. The "most natural" generalization is given by $i)$ and $ii)$ and the following axiom:

jjj) For every $E \in \mathcal{A}$ and $H, K \in \mathcal{K}$

$$Bel(E|K) = Bel(E|H) \cdot [1 - Bel(H^c|K).$$

Nevertheless this axiom does not work, in fact, assuming $jjj)$ we force the conditional belief to be a conditional probability, as the next theorem shows:

Theorem 4 *Let* $\mathcal{H}\{H_i, i = 1, ..., n\}$ *be a set of atoms with* $n \geq 4$ *and* Bel *a function defined on* $\mathcal{C} = \mathcal{A} \times \mathcal{A}^o$, *with* $\mathcal{A}$ = *algebra spanned by* $\mathcal{H}$. *Then* Bel *satisfies condition* $i), ii)$ *and* $jjj)$ *if and only if for every* $K \in \mathcal{A}^o$ $Bel(\dot{|}K)$ *is a probability.*

Proof: We only sketch the proof. Consider any 4 atoms A, B, C, D and put $K = A \vee B \vee C \vee D$. By condition jjj), we have

(*) $Bel(A|A \vee B \vee C) = Bel(A|A \vee B)[1 - Bel(C|A \vee B \vee C)]$;

(**) $Bel(A|K) = Bel(A|A \vee B)[1 - Bel(C \vee D|K)]$;

(***) $Bel(C|K) = Bel(C|A \vee B \vee C)[1 - Bel(D|K)]$.

If $Bel(C|K) + Bel(D|K) = 1$, then trivially follows that $Bel(\cdot|K)$ is a probability. If $Bel(C|K) + Bel(D|K) \neq 1$, then by simple computations we obtain:

$$1 - Bel(C|K) - Bel(D|K)Bel(A|A \vee B) = Bel(A|K)$$

and so, taking into account (**) we obtain

$$Bel(C \vee D|K) = Bel(C|K) + Bel(D|K).$$

With a similar procedure we prove the additivity for all the element of the algebra spanned by K.

On the other hands we easily obtain the assert for $Bel(\cdot|C \vee D)$, by using the last equality and the following ones

$Bel(C|K) = Bel(C|C \vee D)[1 - Bel(A \vee B|K)]$;

$Bel(D|K) = Bel(D|C \vee D)[1 - Bel(A \vee B|K)]$;

$Bel(C \vee D|K) = [1 - Bel(A \vee B|K)]$.

In the literature (see [7]) a different generalization of condition in Definition 5 has been presented. In such a definition the third axiom can be expressed as follows:

jjj') For every $G \in \mathcal{K}$ there exists a number $\underline{Bel}(G|G) \in [0, 1]$ such that for every $E \in \mathcal{A}$ and $H, K \in \mathcal{K}$, with $E \subset H \subset K$ one has

$$Bel(E|H) \odot Bel(H|K) = Bel(E|K)\,.$$

$\odot$ defined as follows

$$Bel(E|H) \odot Bel(H|K) = \begin{cases} 0\,, \quad \text{if} \quad \underline{Bel}(H|H) = 0\,, \\ \frac{Bel(E|H)\,Bel(H|K)}{\underline{Bel}(H|H)}\,, \end{cases} \tag{6}$$

Nevertheless to give a direct definition of $\underline{Bel}(G|G)$ it is necessary to have recourse to the results recalled in section 2.2: in fact, if $P(\cdot|\cdot)$ is a conditional probability defined on the events $(H|H')$, where H, H' are elements of the algebra spanned by the partition related to a belief function, then we have

$$\underline{Bel}(H|H) = P\Big(\bigvee_{H_i \subseteq H} H_i \Big| \bigvee_{H_i H \neq \emptyset} H_i\Big).$$

As proved in [7], the above operation $\odot$ defined in $jjj')$ turns out to be associative and commutative in the set Γ and has 1 as neutral element. Moreover $\odot$ is monotone with respect to the set $\Gamma' \subseteq \Gamma$ of the pairs

$$\Gamma' = \Big\{\Big(Bel(E|H), Bel(H|K)\Big), \Big(Bel(F|H), Bel(H|W)\Big)\Big\}.$$

Moreover it is *distributive* with respect to $\oplus$ in $\mathcal{K}^* \times \Gamma$, where

$$\mathcal{K}^* = \Big\{\Big(Bel(E|H), Bel(F|H)\Big), H \in B, E, F \in \mathcal{G}, EFH = \emptyset\Big\},$$

.

Finally we consider the so called Bayes rule (Def. 4). What we are able to say at this moment is that, if some generalization of this rule in terms of $\odot$-product rule is possible, surely the operation $\odot$ should be not monotone Γ (and also in Γ'. To see that it is sufficient to consider the following simple example:

Example 1 *Let $\mathcal{H} = \{H_1, H_2, H_3\}$ a partition of Ω and $\mathcal{P}$ the class $\{P_1, P_2\}$ with $P_1(H_1) = 2/7$, $P_1(H_2) = 1/7$, $P_1(H_3) = 4/7$ and $P_2(H_1) = 2/7$, $P_2(H_2) = 5/7$, $P_2(H_3) = 0$. It is simple to check that the lower envelope Bel of $\mathcal{P}$ is a belief function on the algebra generated by $\mathcal{H}$. By using as conditional rule Bayes rule, then we have $Bel(H_1) > Bel(H_2)$ but $Bel(H_1|H_1 \vee H_2) < Bel(H_2|H_1 \vee H_2)$.*

By using an algebra spanned by more than 3 atoms it is possible to build chains of conditional events $E|H_i$, $F|H_i$, with $E \subseteq H_1$, $F \subseteq H_1$, and $H_1 \subset H_2 \subset ... \subseteq H_n$ and a (Bayes) conditional belief such that the inequality between $P(E|H_i)$ and $P(F|H_i)$ changes any time we replace H_i with H_{i+1}.

3.1 Characterization of Dempster-conditional Belief

In this section we show, by a characterization theorem, the importance of regarding a conditional belief function as a $(\oplus, \odot)$-decomposable measure. In fact this permits to study the structure underlying the conditional measure and also to build an algorithm to check the consistence (with the model of reference) of a partial assessment (concept known as "coherence"). This kind of characterizations is also the starting point to give more convincing definitions of independence among events with respect to a measure, agreeing with logical independence (see for instance [6], [6] for probability or [9] and [8] for possibility).

Definition 9 *A $Bel(\cdot|\cdot)$ on $\mathcal{C}$ is coherent if there exists $\mathcal{C}' \supset \mathcal{C}$, with $\mathcal{C}' = \mathcal{E} \times \mathcal{H}^o$ ($\mathcal{E}$ a Boolean algebra, $\mathcal{H}$ an additive set), such that $P(\cdot|\cdot)$ can be extended from $\mathcal{C}$ to $\mathcal{C}'$ as a conditional Belief.*

The following theorem characterizes (coherent) Dempster-conditional Belief in terms of a class of Plausibilities $\{Pl_1, ..., Pl_m\}$ with Pl_k defined on the additive set generated by the conditioning events with $Pl_{k-1} = 0$, agreeing in the sense of Definition 7.

Theorem 5 *Let $\mathcal{F} = \{E_1|F_1, E_2|F_2, \ldots, E_m|F_m\}$ be a finite family of conditional events and denote by $\mathcal{K} = \{H_1, H_2, ..., H_n\}$ the algebra spanned by $\{E_1, \ldots, E_m, F_1, \ldots, F_m\}$. For a real function Bel on $\mathcal{F}$ the following statements are equivalent:*

(a) $Bel : \mathcal{F} \to [0,1]$ is a coherent Dempster conditional belief assessment;

(b) There exists (at least) a class $\{Pl_\alpha, Bel_\alpha\}$ of plausibility functions and relevant dual belief functions such that, called H_0^α the greatest set of $\mathcal{K}$ for which $Pl_{(\alpha-1)}(H_0^\alpha) = 0$, we have $Pl_\alpha(H_0^\alpha) = 1$ and $H_0^\alpha \subset H_0^\beta$ for all $\beta < \alpha$.

Moreover, for every $E_i|F_i$, there exists a β such that, $Pl_\alpha(F_i) = 0$ for all $\alpha < \beta$, $Pl_\beta(F_i) > 0$ and

$$Bel(E_i|F_i) = 1 - \frac{Pl_\beta(E_i^c F_i)}{Pl_\beta(F_i)}, \tag{7}$$

(c) all the following systems (S^α), with $\alpha = 0, 1, 2, ...k \leq n$, admit a solution $\mathbf{X}^\alpha = x_k^\alpha = m_\alpha(H_k)$:

$$(S^\alpha) = \begin{cases} \sum\limits_{H_k F_i \neq \emptyset} x_k^\alpha \cdot [1 - Bel(E_i|F_i)] = \sum\limits_{H_k G_i^c F_i \neq \emptyset} x_k^\alpha, & \forall F_i \subseteq H_1^\alpha \\ \sum\limits_{H_k \in H_0^\alpha} x_k^\alpha = 1; & \\ x_k^\alpha \geq 0, & \forall H_k \subseteq H_1^\alpha; \end{cases}$$

where $G_i = E_i F_i$ and $H_1^0 = H_1 \vee H_2 \vee \ldots \vee H_n$,
$H_1^\alpha = \bigvee\limits_{H_i H_0^\alpha \neq \emptyset} H_i$ and H_0^α is the maximum element of $\mathcal{K}$ (with respect to $\subseteq$) such that $\sum\limits_{H_i H_0^\alpha \neq \emptyset} m_{(\alpha-1)}(H_i) = 0$.

References

[1] B. Bouchon-Meunier, G. Coletti, C. Marsala (2001). Conditional Possibility and Necessity. In *Technologies for Constructing Intelligent Systems (B. Bouchon-Meunier, J. Gutiérrez-Rios, L. Magdalena, R.R. Yager eds.)* volume 1, Springer, Berlin, 2001 (Selected papers from IPMU 2000, Madrid).

[2] B. Bouchon-Meunier, G. Coletti, C. Marsala (2002) Independence and Possibilistic Conditioning. *Annals of Mathematics and Artificial Intelligence*, 35, pp. 107-124.

[3] G. Coletti, R. Scozzafava. Conditioning and Inference in Intelligent Systems. *Soft Computing*, 3 (3) (1999) 118–130

[4] Coletti G, Scozzafava R. (2001) From conditional events to conditional measures: a new axiomatic approach. *Annals of Mathematics and Artificial Intelligence* , 32, 373–392.

[5] Coletti G, Scozzafava R. (2002), Stochastic Independence in a Coherent Setting, Annals of Mathematics and Artificial Intelligence 35, 151176.

[6] Coletti G, Scozzafava R. Probabilistic Logic in a Coherent Setting (Trends in Logic, n.15). Dordrecht: Kluwer: 2002.

[7] Coletti G, Scozzafava R. (2006) Toward a General Theory of Conditional Beliefs. *International Journal of Intelligent Systems*, vol. 21, 229-259.

[8] Coletti G., Vantaggi B. (2006) Possibility theory: conditional independence. *Fuzzy Sets and Systems*, in press

[9] Ferracuti L., Vantaggi B. (2006) Independence and conditional possibilities for strictly monotone triangular norms. *International Journal of Intelligent Systems*, vol. 21, 21(3), 299-323.

[10] R.T. Cox. Probability, frequency and reasonable expectation. *American Journal of Physics*, 14 (1) (1946) 1–13

[11] B. de Finetti. La logique de la probabilité. *Actes du Congrès International de Philosophie Scientifique*, Paris, Hermann: IV, 1–9, 1935

[12] B. de Finetti. Sull'impostazione assiomatica del calcolo delle probabilità. *Annali Univ. Trieste*, 19 (1949) 3–55 - *Engl. transl.*: Ch.5 in *Probability, Induction, Statistics*, Wiley, London, 1972.

[13] A.P. Dempster (1968) A generalizatin of Bayesian Inference. *The Royal Stat. Soc. B*, 50, 205–247.

[14] J.Y. Halpern (1999) A counterexample to theorems of Cox and Fine. *J. of Artificial Intelligence Research*, 10, 67–85.

[15] J.Y. Jaffray (1992) Bayesian updatingand Belief functions. *IEEE Trans. on Systems, man and Cybernetic*, 5, 1144–1152.

[16] Popper, K.R.: The Logic of Scientific Discovery. Routledge, London and New York (1959)

[17] A. Rényi. On conditional probability spaces generated by a dimensionally ordered set of measures. *Theor. Probab. Appl.*, 1 (1956) 61–71

[18] P. Smets (2000) Belief function and transferable belief model. *Documentation of SIPTA*, www.sipta.org/documentation/belief/belief.ps.

[19] P. Walley (1987) Belief function representations of statistical evidence. *Ann. of Stat.*, 4, 1439–1465. Independence and