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QUASICONVEXITY AT THE BOUNDARY AND CONCENTRATION EFFECTS GENERATED BY GRADIENTS *

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Abstract. We characterize generalized Young measures, the so-called DiPerna–Majda measures which are generated by sequences of gradients. In particular, we precisely describe these measures at the boundary of the domain in the case of the compactification of $\mathbb{R}^{m\times n}$ by the sphere. We show that this characterization is closely related to the notion of quasiconvexity at the boundary introduced by Ball and Marsden [J.M. Ball and J. Marsden, *Arch. Ration. Mech. Anal.* **86** (1984) 251–277]. As a consequence we get new results on weak $W^{1,2}(\Omega;\mathbb{R}^3)$ sequential continuity of $u\mapsto a\cdot[\mathrm{Cof}\,\nabla u]\varrho$, where $\Omega\subset\mathbb{R}^3$ has a smooth boundary and a,ϱ are certain smooth mappings.

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1. Introduction

Oscillations and/or concentrations appear in many problems in the calculus of variations, partial differential equations, or optimal control theory, which admit only L^p but not L^∞ apriori estimates. While Young measures [41] successfully capture oscillatory behavior (see *e.g.* [21, 28, 31, 32]) of sequences they completely miss concentrations. There are several tools how to deal with concentrations. They can be considered as generalization of Young measures, see for example Alibert's and Bouchitté's approach [1], DiPerna's and Majda's treatment of concentrations [6], or Fonseca's method described in [10]. An overview can be found in [30, 38]. Moreover, in many cases, we are interested in oscillation/concentration effects generated by sequences of gradients. A characterization of Young measures generated by gradients was completely given by Kinderlehrer and Pedregal [16, 18], *cf.* also [28, 29]. The first attempt to characterize both oscillations and concentrations in sequences of gradients is due to Fonseca *et al.* [12]. They dealt with a special situation of $\{gv(\nabla u_k)\}_{k\in\mathbb{N}}$ where v coincides with a positively p-homogeneous function at infinity (see (1.35) for a precise statement), $u_k\in W^{1,p}(\Omega;\mathbb{R}^m)$, $p>1$, with g continuous and vanishing on $\partial\Omega$. Later on, a characterization of oscillation/concentration effects in terms of DiPerna's and Majda's generalization of Young measures was given in [15] for arbitrary integrands and in [11] for sequences living in the kernel of a first-order differential operator. Recently Kristensen and Rindler [19] characterized oscillation/concentration effects in the case $p=1$. Nevertheless, a complete analysis of boundary

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effects generated by gradients is still missing. We refer to [15] for the case where $u_k = u$ on the boundary of the domain. As already observed by Meyers [24], concentration effects at the boundary are closely related to the sequential weak lower semicontinuity of integral functionals $I : W^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}$: $I(u) = \int_\Omega v(\nabla u(x))\,\mathrm{d}x$ where $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is continuous and such that $|v| \le C(1 + |\cdot|^p)$ for some constant $C > 0$, *cf.* also [20] for recent results. Indeed, consider $u \in W^{1,p}_0(B(0,1);\mathbb{R}^m)$, where $B(0,1)$ is the unit ball in $\mathbb{R}^n$ centered at 0, and extend it by zero to the whole $\mathbb{R}^n$. Define for $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$ $u_k(x) = k^{n/p-1}u(kx)$, *i.e.*, $u_k \rightharpoonup 0$ in $W^{1,p}(B(0,1);\mathbb{R}^m)$ and consider a smooth convex domain $\Omega \in \mathbb{R}^n$ such that $0 \in \partial\Omega$, ϱ is the outer unit normal to $\partial\Omega$ at 0 and let there be $x \in \Omega$ such that $\varrho\cdot x < 0$. Moreover, take a function v to be positively p-homogeneous, *i.e.*, $v(\alpha s) = \alpha^p v(s)$ for all $\alpha \ge 0$. Then if I is weakly lower semicontinuous then

$$0 = I(0) \le \liminf_{k\to\infty}\int_\Omega v(\nabla u_k(x))\,\mathrm{d}x = \liminf_{k\to\infty}\int_{B(0,1)\cap\Omega} v(\nabla u_k(x))\,\mathrm{d}x = \liminf_{k\to\infty}\int_{B(0,1)\cap\Omega} k^n v(\nabla(u(kx))\,\mathrm{d}x$$
$$= \int_{B(0,1)\cap\{x\in\mathbb{R}^n;\ \varrho\cdot x<0\}} v(\nabla u(y))\,\mathrm{d}y. \tag{1.1}$$

Thus, we see that

$$0 \le \int_{B(0,1)\cap\{x\in\mathbb{R}^n;\ \varrho\cdot x<0\}} v(\nabla u(y))\,\mathrm{d}y$$

for all $u \in W^{1,p}_0(B(0,1);\mathbb{R}^m)$ forms a necessary condition for weak lower semicontinuity of I. Here we show that the weak lower semicontinuity of the above defined functional I is intimately related to the so-called *quasiconvexity at the boundary* defined by Ball and Marsden in [3] and that this notion of quasiconvexity plays a crucial role in the characterization of parametrized measures generated by sequences of gradients. Moreover, we show that if $\{u_k\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$, $u_k \rightharpoonup u$, and $h(x,s) := [\mathrm{Cof}\, s]\cdot(a(x)\otimes\varrho(x))$ ("Cof" denotes the cofactor matrix) for some $a,\varrho \in C(\bar\Omega;\mathbb{R}^3)$ such that ϱ coincides with the outer unit normal to $\partial\Omega$ on the boundary $\partial\Omega$ of a smooth bounded domain $\Omega \subset \mathbb{R}^3$ then $h(\cdot,\nabla u_k) \to h(\cdot,\nabla u)$ weakly* in Radon measures supported in $\bar\Omega$. If, additionally, $h(x,\nabla u_k(x)) \ge 0$ for all $k \in \mathbb{N}$ and almost all $x \in \Omega$ then the above convergence is even in the weak topology of $L^1(\Omega)$. Hence, there is a continuous function $\psi : [0,+\infty) \to [0,+\infty)$ such that $\lim_{t\to\infty}\psi(t)/t = +\infty$ and $\sup_{k\in\mathbb{N}}\int_\Omega \psi\,(h(x,\nabla u_k(x))\,\mathrm{d}x < +\infty$. This result, which can be generalized to higher dimensions, too, is an analogy to the celebrated S. Müler's result on higher integrability of determinants [27]. See also [14, 17].

1.1. Basic notation

Let us start with a few definitions and with the explanation of our notation. Having a bounded domain $\Omega \subset \mathbb{R}^n$ we denote by $C(\Omega)$ the space of continuous functions: $\Omega \to \mathbb{R}$. Then $C_0(\Omega)$ consists of functions from $C(\Omega)$ whose support is contained in Ω. In what follows "rca(S)" denotes the set of regular countably additive set functions on the Borel σ-algebra on a metrizable set S (*cf.* [7]), its subset, $\mathrm{rca}^+_1(S)$, denotes regular probability measures on a set S. We write "γ-almost all" or "γ-a.e." if we mean "up to a set with the γ-measure zero". If γ is the n-dimensional Lebesgue measure and $M \subset \mathbb{R}^n$ we omit writing γ in the notation. Further, $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 \le p < +\infty$ denotes the usual space of measurable mappings which are together with their first (distributional) derivatives integrable with the p-th power. The support of a measure $\sigma \in$ rca(Ω) is a smallest closed set S such that $\sigma(A) = 0$ if $S \cap A = \emptyset$. Finally, if $\sigma \in \mathrm{rca}(S)$ we write σ_s and d_σ for the singular part and density of σ defined by the Lebesgue decomposition, respectively. We denote by 'w-lim' the weak limit and by $B(x_0,r)$ an open ball in $\mathbb{R}^n$ centered at x_0 and the radius $r > 0$. The dot product on $\mathbb{R}^n$ is standardly defined as $a\cdot b := \sum_{i=1}^n a_ib_i$ and analogously on $\mathbb{R}^{m\times n}$. Finally, if $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ then $a\otimes b \in \mathbb{R}^{m\times n}$ with $(a\otimes b)_{ij} = a_ib_j$, and $\mathbb{I}$ denotes the identity matrix.

If not said otherwise, we will suppose in the sequel that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a C^1 boundary. The same regularity is assumed if we say that Ω has a smooth boundary.

1.2. Quasiconvex functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We say that a function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex [26] if for any $s_0 \in \mathbb{R}^{m\times n}$ and any $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$

$$v(s_0)|\Omega| \le \int_\Omega v(s_0 + \nabla\varphi(x))\,\mathrm{d}x. \tag{1.2}$$

If $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is not quasiconvex we define its quasiconvex envelope $Qv : \mathbb{R}^{m\times n} \to \mathbb{R}$ as

$$Qv = \sup\{h \le v;\ h : \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex }\}$$

and if the set on the right-hand side is empty we put $Qv = -\infty$. If v is locally bounded and Borel measurable then for any $s_0 \in \mathbb{R}^{m\times n}$ (see [5])

$$Qv(s_0) = \inf_{\varphi\in W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \frac{1}{|\Omega|}\int_\Omega v(s_0 + \nabla\varphi(x))\,\mathrm{d}x. \tag{1.3}$$

We will also need the following elementary result. It can be found in a more general form *e.g.* in [5], Chapter 4, Lemma 2.2, or in [26].

Lemma 1.1. *Let $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ be quasiconvex with $|v(s)| \le C(1 + |s|^p)$, $C > 0$, for all $s \in \mathbb{R}^{m\times n}$. Then there is a constant $\alpha \ge 0$ such that for every $s_1, s_2 \in \mathbb{R}^{m\times n}$ it holds*

$$|v(s_1) - v(s_2)| \le \alpha(1 + |s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|. \tag{1.4}$$

Following [3, 34, 36] we define the notion of quasiconvexity at the boundary. In order to proceed, we first define the so-called *standard boundary domain*.

Definition 1.2. Let $\varrho \in \mathbb{R}^n$ be a unit vector and let Ω_ϱ be a bounded open Lipschitz domain. We say that Ω_ϱ is a standard boundary domain with the normal ϱ if there is $a \in \mathbb{R}^n$ such that $\Omega_\varrho \subset H_{a,\varrho} := \{x \in \mathbb{R}^n;\ \varrho\cdot x < a\}$ and the $(n-1)$- dimensional interior Γ_ϱ of $\partial\Omega_\varrho \cap \partial H_{a,\varrho}$ is nonempty.

We are now ready to define the quasiconvexity at the boundary. We put for $1 \le p \le +\infty$

$$W^{1,p}_{\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m) := \{u \in W^{1,p}(\Omega_\varrho;\mathbb{R}^m);\ u = 0 \text{ on } \partial\Omega_\varrho \setminus \Gamma_\varrho\}. \tag{1.5}$$

Definition 1.3. ([3]) Let $\varrho \in \mathbb{R}^n$ be a unit vector. A function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called quasiconvex at the boundary at $s_0 \in \mathbb{R}^{m\times n}$ with respect to ϱ (shortly v is qcb at (s_0,ϱ)) if there is $q \in \mathbb{R}^m$ such that for all $u \in W^{1,\infty}_{\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$ it holds

$$\int_{\Gamma_\varrho} q\cdot u(x)\,\mathrm{d}S + v(s_0)|\Omega_\varrho| \le \int_{\Omega_\varrho} v(s_0 + \nabla u(x))\,\mathrm{d}x. \tag{1.6}$$

An immediate generalization is the following.

Definition 1.4. Let $\varrho \in \mathbb{R}^n$ be a unit vector, $1 \le p < +\infty$. A function $v : \mathbb{R}^{m\times n} \to \mathbb{R}$, $|v| \le C(1 + |\cdot|^p)$ for some $C > 0$ is called $W^{1,p}$-quasiconvex at the boundary at $s_0 \in \mathbb{R}^{m\times n}$ with respect to ϱ (shortly v is p-qcb at (s_0,ϱ)) if there is $q \in \mathbb{R}^m$ such that for all $u \in W^{1,p}_{\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$ it holds

$$\int_{\Gamma_\varrho} q\cdot u(x)\,\mathrm{d}S + v(s_0)|\Omega_\varrho| \le \int_{\Omega_\varrho} v(s_0 + \nabla u(x))\,\mathrm{d}x. \tag{1.7}$$

Remark 1.5.

(i) If v is differentiable then $q := \frac{\partial v}{\partial s}(s_0)\varrho$ is given uniquely; *cf.* [36]. We denote the set of such of vectors q for which (1.6) holds by $\partial_v^{\rm qcb}(s_0,\varrho)$. It may be seen as a notion of a "subdifferential" for v.
(ii) It is clear that if v is qcb at (s_0,ϱ) it is also quasiconvex at s_0, *i.e.*, (1.2) holds.
(iii) If (1.6) holds for one standard boundary domain it holds for other standard boundary domains, too.
(iv) If $p>1$, $v:\mathbb{R}^{m\times n}\to\mathbb{R}$ is positively p-homogeneous, *i.e.* $v(\lambda s)=\lambda^p v(s)$ for all $s\in\mathbb{R}^{m\times n}$, continuous, and p-qcb at $(0,\varrho)$ then $q=0$ in (1.6). Indeed, we have $v(0)=0$ and suppose that $\int_{\Omega_\varrho} v(\nabla u(x))\,{\rm d}x<0$ for some $u\in W^{1,\infty}_{\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$. By (1.6), we must have for all $\lambda>0$

$$0\le \lambda^p\int_{\Omega_\varrho} v(\nabla u(x))\,{\rm d}x-\lambda\int_{\Gamma_\varrho} q\cdot u(x)\,{\rm d}S.$$

However, it is not possible for $\lambda>0$ large enough and therefore for all $u\in W^{1,\infty}_{\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$ it holds that $\int_{\Omega_\varrho} v(\nabla u(x))\,{\rm d}x\ge 0$. Thus, we can take $q=0$.

The following lemma shows that Definitions 1.3 and 1.4 are equivalent for a class of functions whose modulus grows as the p-th power.

Lemma 1.6. *Let $v:\mathbb{R}^{m\times n}\to\mathbb{R}$ be continuous and such that $|v(A)|\le C(1+|A|^p)$ for all $A\in\mathbb{R}^{m\times n}$ and some $C>0$ independent of A and some $1\le p<+\infty$. If v is qcb at (s_0,ϱ) it is p-qcb at (s_0,ϱ).*

Proof. Take $u\in W^{1,p}_{\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$ and a sequence $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,\infty}_{\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$ such that $u_k\to u$ strongly in $W^{1,p}(\Omega_\varrho;\mathbb{R}^m)$. We get using (1.4) that

$$\int_{\Omega_\varrho} v(s_0+\nabla u(x))\,{\rm d}x=\lim_{k\to\infty}\int_{\Omega_\varrho} v(s_0+\nabla u_k(x))\,{\rm d}x. \tag{1.8}$$

As v is qcb at (s_0,ϱ) we have

$$\lim_{k\to\infty}\int_{\Omega_\varrho} v(s_0+\nabla u_k(x))\,{\rm d}x\ge|\Omega_\varrho|v(s_0)+\lim_{k\to\infty}\int_{\Gamma_\varrho} q\cdot u_k(x)\,{\rm d}S=|\Omega_\varrho|v(s_0)+\int_{\Gamma_\varrho} q\cdot u(x)\,{\rm d}S, \tag{1.9}$$

which finishes the proof in view of (1.8). □

It will be convenient to define the following notion recalling the quasiconvex envelope of v at zero. Here, however, we integrate only over a standard boundary domain with a given normal. If $\varrho\in\mathbb{R}^n$ has a unit length then put

$$Q_{b,\varrho}v(0):=\inf_{u\in W^{1,p}_{\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)}\frac{1}{|\Omega_\varrho|}\int_{\Omega_\varrho} v(\nabla u(x))\,{\rm d}x. \tag{1.10}$$

Remark 1.7. If v is positively p homogeneous with $p>1$ then either $Q_{b,\varrho}v(0)=0$ or $Q_{b,\varrho}v(0)=-\infty$. We also have that $Q_{b,\varrho}v(0)\le Qv(0)$. Having a ball $B(0,1)=\{x\in\mathbb{R}^n;\ |x|<1\}$ we put $\Omega_\varrho:=B(0,1)\cap\{x\in\mathbb{R}^n;\ \varrho\cdot x<0\}$. In this case, we only integrate over a half-ball in (1.10). Hence, we can use only those $u\in W^{1,p}_{\Gamma_\varrho}(\Omega_\varrho;\mathbb{R}^m)$ which are symmetric with respect to the plane $\{x;\ \varrho\cdot x=0\}$, *i.e.*, satisfying $u(x)=u(x-2(\varrho\cdot x)\varrho)$ if $x\in\Omega_\varrho$.

Quasiconvexity at the boundary was introduced in [3] as a necessary condition for strong local minima of the mixed problem in nonlinear elasticity at boundary points beloging to a free part of the boundary. Contrary to the usual Morrey's quasiconvexity there are not many papers dealing with this notion. Let me point out several interesting results in this direction. Mielke and Sprenger [25] investigated relation of quasiconvexity at the boundary and Agmon's condition for quadratic stored energies in nonlinear elasticity and Sprenger [36] in his thesis defined the so-called polyconvexity at the boundary. Recently, Grabovsky and Mengesha [13] showed that quasiconvexity at the boundary is a sufficient condition for the so-called $W^{1,\infty}$-sequential weak* local minima - slight weakening of the notion of strong local minimizers. Here we find another interesting connection, namely the fact that quasiconvexity at the boundary plays a crucial role in the analysis of concentration effects generated by gradients and is essential for $W^{1,p}$-sequential weak lower semicontinuity of integral functionals as in (1.1).

We start with the following auxiliary lemma.

Lemma 1.8. *Let $v:\mathbb{R}^{m\times n}\to\mathbb{R}$, $|v|\le C(1+|\cdot|^p)$, $C>0$, $1\le p<+\infty$, be quasiconvex, and such that $v(0)=Q_{b,\varrho}v(0)=0$ for some $\varrho\in\mathbb{R}^n$. Let Ω be a bounded domain in $\mathbb{R}^n$ with the C^1 boundary, with $x_0\in\partial\Omega$, and with ϱ the outer unit normal at x_0. Then for every $\varepsilon>0$ there is $\delta>0$ and a continuous function $f:\mathbb{R}\to(0+\infty)$, $\lim_{\varepsilon\to 0}\varepsilon f(\varepsilon)=0$, such that $\Omega\cap B(x_0,\delta)\subset\Omega$ and it holds that for every $U\in W_0^{1,p}(B(0,\delta);\mathbb{R}^m)$ that*

$$\int_{B(x_0,\delta)\cap\Omega} v(\nabla U(x))\,\mathrm{d}x \ge -\varepsilon\int_{B(x_0,\delta)\cap\Omega}(|\nabla U(x)|+f(\varepsilon)|\nabla U(x)|^p)\,\mathrm{d}x, \tag{1.11}$$

Proof. Following [3], we can assume, without loss of generality, that $x_0=0$ and that $\varrho=(0,0,\dots,0,1)\in\mathbb{R}^n$. Let further

$$\partial\Omega\cap B(0,r):=\{x\in B(0,r);\ x_n=h(x')\},$$
$$\Omega\cap B(0,r):=\{x\in B(0,r);\ x_n<h(x')\},$$

where $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, $x'=(x_1,\dots,x_{n-1})$, and $h\in C^1(\mathbb{R}^{n-1})$ is such that $h(0)=0$ and $\nabla h(0)=0$. As in [3] we define for $\xi>0$ $X_\xi:\mathbb{R}^n\to\mathbb{R}^n$ by $X_\xi(y)=\xi y+h(\xi y')\varrho$. Notice that $\nabla X_\xi(y)=\xi(\mathbb{I}+\varrho\otimes\nabla h(\xi y'))$ and that $\det\nabla X_\xi=\xi^n$ because $\varrho\cdot\nabla h=0$. Let $U\in W_0^{1,p}(X_\xi(B(0,r);\mathbb{R}^m)$. Define $u(y):=\frac{1}{\xi}U(X_\xi(y))$, *i.e.* $u\in W_0^{1,p}(B(0,r);\mathbb{R}^m)$. Then $\nabla u(y)=\frac{1}{\xi}\nabla U(X_\xi(y))\nabla X_\xi(y)=\frac{1}{\xi}\nabla U(z)\nabla X_\xi(X_\xi^{-1}(z))$ for $z:=X_\xi(y)$. Notice that $X_\xi^{-1}(z)=\xi^{-1}(z-h(z')\varrho)$ for all $z\in\mathbb{R}^n$. For $\Omega_\varrho:=\{x\in B(0,r);\ x_n<0\}$, we calculate using Lemma 1.1

$$\begin{aligned}\int_{X_\xi(\Omega_\varrho)} v(\nabla U(z))\mathrm{d}z+\alpha\int_{X_\xi(\Omega_\varrho)}\left(1+|\nabla U(z)|^{p-1}+\left|\frac{1}{\xi}\nabla U(z)\nabla X_\xi(X_\xi^{-1}(z))\right|^{p-1}\right)\left|\nabla U(z)\left(\mathbb{I}-\frac{1}{\xi}\nabla X_\xi(X_\xi^{-1}(z)\right)\right|\mathrm{d}z \\ \ge\int_{X_\xi(\Omega_\varrho)} v\left(\frac{1}{\xi}\nabla U(z)\nabla X_\xi(X_\xi^{-1}(z))\right)\mathrm{d}z=\xi^n\int_{\Omega_\varrho} v(\nabla u(y))\,\mathrm{d}y\ge 0.\end{aligned} \tag{1.12}$$

The last inequality follows from the assumption $Q_{b,\varrho}v(0)\ge 0$. Hence, exploiting the identity $\xi^{-1}\nabla X_\xi(X_\xi^{-1}(z))=\mathbb{I}+\varrho\otimes\nabla h(z')$, we get

$$\begin{aligned}\int_{X_\xi(\Omega_\varrho)} v(\nabla U(z))\mathrm{d}z\ge-\alpha\int_{X_\xi(\Omega_\varrho)}\left(1+|\nabla U(z)|^{p-1}+\left|\frac{1}{\xi}\nabla U(z)\nabla X_\xi(X_\xi^{-1}(z))\right|^{p-1}\right)\left|\nabla U(z)\left(\mathbb{I}-\frac{1}{\xi}\nabla X_\xi(X_\xi^{-1}(z)\right)\right|\mathrm{d}z \\ =-\alpha\int_{X_\xi(\Omega_\varrho)}(|\nabla U(z)|+|\nabla U(z)|^p(1+|\mathbb{I}+\varrho\otimes\nabla h(z')|^{p-1}))|\varrho\otimes\nabla h(z')|\,\mathrm{d}z.\end{aligned} \tag{1.13}$$

However,

$$\begin{aligned}0&\le\int_{X_\xi(\Omega_\varrho)}(|\nabla U(z)|+|\nabla U(z)|^p(1+|\mathbb{I}+\varrho\otimes\nabla h(z')|^{p-1}))|\varrho\otimes\nabla h(z')|\,\mathrm{d}z \\ &\le\int_{X_\xi(\Omega_\varrho)}(|\nabla U(z)|+|\nabla U(z)|^p(1+2^{p-1}(n^{(p-1)/2}+M(|z'|)^{p-1})))M(|z'|)\,\mathrm{d}z,\end{aligned} \tag{1.14}$$

where M is the modulus of continuity of the uniformly continuous function $z \mapsto \varrho \otimes \nabla h(z')$ on $\overline{X_\xi(\Omega_\varrho)}$, *i.e.*, $\lim_{s\to 0_+} M(s) = 0$. Thus, choosing $\varepsilon > 0$, we take $\xi > 0$ so small that $\sup_{z\in X_\xi(\Omega_\varrho)} M(|z|) < \varepsilon/\alpha$ and define $f(\varepsilon) := 1 + 2^{p-1}(n^{(p-1)/n} + (\varepsilon/\alpha)^{p-1})$. Then we have

$$\begin{aligned}0 &\le \alpha \int_{X_\xi(\Omega_\varrho)} (|\nabla U(z)| + |\nabla U(z)|^p(1 + 2^{p-1}(n^{(p-1)/2} + M(|z'|)^{p-1})))M(|z'|)\,\mathrm{d}z\\ &\le \int_{X_\xi(\Omega_\varrho)} (|\nabla U(z)| + f(\varepsilon)|\nabla U(z)|^p)\varepsilon\,\mathrm{d}z.\end{aligned}$$

Finally, in view of (1.13) we have

$$\int_{X_\xi(\Omega_\varrho)} v(\nabla U(z))\,\mathrm{d}z \ge -\varepsilon \int_{X_\xi(\Omega_\varrho)} (|\nabla U(z)| + f(\varepsilon)|\nabla U(z)|^p)\,\mathrm{d}z.$$

We take $\delta > 0$ such that $B(0,\delta) \cap \Omega \subset X_\xi(\Omega_\varrho)$ and take $U \in W^{1,p}_0(B(0,\delta);\mathbb{R}^m)$ extended to $\mathbb{R}^n$ by zero which is admissible. Then, as $v(0) = 0$, we have from the previous inequality that

$$\int_{B(0,\delta)\cap\Omega} v(\nabla U(z))\,\mathrm{d}z \ge -\varepsilon \int_{B(0,\delta)\cap\Omega} (|\nabla U(z)| + f(\varepsilon)|\nabla U(z)|^p)\,\mathrm{d}z. \qquad \square$$

Example 1.9. If $n = m = 3$ then it is shown in [34], Proposition 17.2.4, that the function $v : \mathbb{R}^{n\times n} \to \mathbb{R}$ given by

$$v(s) = a \cdot [\mathrm{Cof}s]\varrho$$

is quasiconvex at the boundary with the unit normal $\varrho \in \mathbb{R}^n$. Here $a \in \mathbb{R}^n$ is an arbitrary constant and "Cof" is the cofactor matrix, *i.e.*, $\mathrm{Cof}s_{ij} = (-1)^{i+j}\mathrm{det}s'_{ij}$, where s'_{ij} is the submatrix of s obtained from s by removing the i-th row and the j-th column. Hence, v is positively 2-homogeneous. See also [35] for the role of this v in the definition of the so-called interface polyconvexity.

1.3. Young measures

For $p \ge 0$ we define the following subspace of the space $C(\mathbb{R}^{m\times n})$ of all continuous functions on $\mathbb{R}^{m\times n}$:

$$C_p(\mathbb{R}^{m\times n}) = \{v \in C(\mathbb{R}^{m\times n}); v(s) = o(|s|^p) \text{ for } |s| \to \infty\}.$$

The Young measures on a bounded domain $\Omega \subset \mathbb{R}^n$ are weakly* measurable mappings $x \mapsto \nu_x : \Omega \to \mathrm{rca}(\mathbb{R}^{m\times n})$ with values in probability measures; and the adjective "weakly* measurable" means that, for any $v \in C_0(\mathbb{R}^{m\times n})$, the mapping $\Omega \to \mathbb{R} : x \mapsto \langle \nu_x, v\rangle = \int_{\mathbb{R}^{m\times n}} v(\lambda)\nu_x(\mathrm{d}\lambda)$ is measurable in the usual sense. Let us remind that, by the Riesz theorem, $\mathrm{rca}(\mathbb{R}^{m\times n})$, normed by the total variation, is a Banach space which is isometrically isomorphic with $C_0(\mathbb{R}^{m\times n})^*$, where $C_0(\mathbb{R}^{m\times n})$ stands for the space of all continuous functions $\mathbb{R}^{m\times n} \to \mathbb{R}$ vanishing at infinity. Let us denote the set of all Young measures by $\mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$. It is known that $\mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ is a convex subset of $L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{m\times n})) \cong L^1(\Omega;C_0(\mathbb{R}^{m\times n}))^*$, where the subscript "w" indicates the property "weakly* measurable". A classical result [37, 40] is that, for every sequence $\{y_k\}_{k\in\mathbb{N}}$ bounded in $L^\infty(\Omega;\mathbb{R}^{m\times n})$, there exists its subsequence (denoted by the same indices for notational simplicity) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\forall v \in C_0(\mathbb{R}^{m\times n}) : \qquad \lim_{k\to\infty} v \circ y_k = v_\nu \qquad \text{weakly* in } L^\infty(\Omega), \tag{1.15}$$

where $[v \circ y_k](x) = v(y_k(x))$ and

$$v_\nu(x) = \int_{\mathbb{R}^{m\times n}} v(\lambda)\nu_x(\mathrm{d}\lambda). \tag{1.16}$$

Let us denote by $\mathcal{Y}^\infty(\Omega;\mathbb{R}^{m\times n})$ the set of all Young measures which are created by this way, *i.e.* by taking all bounded sequences in $L^\infty(\Omega;\mathbb{R}^{m\times n})$. Note that (1.15) actually holds for any $v:\mathbb{R}^{m\times n}\to\mathbb{R}$ continuous.

A generalization of this result was formulated by Schonbek [33] (*cf.* also [2]): if $1\le p<+\infty$: for every sequence $\{y_k\}_{k\in\mathbb{N}}$ bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ there exists its subsequence (denoted by the same indices) and a Young measure $\nu=\{\nu_x\}_{x\in\Omega}\in\mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\forall v\in C_p(\mathbb{R}^{m\times n}):\qquad \lim_{k\to\infty} v\circ y_k = v_\nu \qquad \text{weakly in } L^1(\Omega). \tag{1.17}$$

We say that $\{y_k\}$ generates ν if (1.17) holds.

Let us denote by $\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ the set of all Young measures which are created by this way, *i.e.* by taking all bounded sequences in $L^p(\Omega;\mathbb{R}^{m\times n})$.

The following important lemma was proved in [12].

Lemma 1.10. *Let $1<p<+\infty$ and $\Omega\subset\mathbb{R}^n$ be an open bounded set and let $\{u_k\}_{k\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ be bounded. Then there is a subsequence $\{u_j\}_{j\in\mathbb{N}}$ and a sequence $\{z_j\}_{j\in\mathbb{N}}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$\lim_{j\to\infty} |\{x\in\Omega;\ z_j(x)\neq u_j(x) \text{ or } \nabla z_j(x)\neq\nabla u_j(x)\}| = 0 \tag{1.18}$$

and $\{|\nabla z_j|^p\}_{j\in\mathbb{N}}$ is relatively weakly compact in $L^1(\Omega)$. In particular, $\{\nabla u_j\}$ and $\{\nabla z_j\}$ generate the same Young measure.

1.4. DiPerna–Majda measures

Let us take a complete (*i.e.* containing constants, separating points from closed subsets and closed with respect to the Chebyshev norm) separable ring $\mathcal{R}$ of continuous bounded functions $\mathbb{R}^{m\times n}\to\mathbb{R}$. It is known [8], Section 3.12.21, that there is a one-to-one correspondence $\mathcal{R}\mapsto\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$ between such rings and metrizable compactifications of $\mathbb{R}^{m\times n}$; by a compactification we mean here a compact set, denoted by $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$, into which $\mathbb{R}^{m\times n}$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^{m\times n}$ and its image in $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$. Similarly, we will not distinguish between elements of $\mathcal{R}$ and their unique continuous extensions on $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$.

Let $\sigma\in\mathrm{rca}(\bar\Omega)$ be a positive Radon measure on a bounded domain $\Omega\subset\mathbb{R}^n$. A mapping $\hat\nu: x\mapsto\hat\nu_x$ belongs to the space $L^\infty_{\mathrm{w}}(\bar\Omega,\sigma;\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ if it is weakly* σ-measurable (*i.e.*, for any $v_0\in C_0(\mathbb{R}^{m\times n})$, the mapping $\bar\Omega\to\mathbb{R}: x\mapsto\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)$ is σ-measurable in the usual sense). If additionally $\hat\nu_x\in\mathrm{rca}_1^+(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ for σ-a.a. $x\in\bar\Omega$ the collection $\{\hat\nu_x\}_{x\in\bar\Omega}$ is the so-called Young measure on $(\bar\Omega,\sigma)$ [41], see also [2, 30, 37, 39, 40].

DiPerna and Majda [6] shown that having a bounded sequence in $L^p(\Omega;\mathbb{R}^{m\times n})$ with $1\le p<+\infty$ and Ω an open domain in $\mathbb{R}^n$, there exists its subsequence (denoted by the same indices), a positive Radon measure $\sigma\in\mathrm{rca}(\bar\Omega)$, and a Young measure $\hat\nu: x\mapsto\hat\nu_x$ on $(\bar\Omega,\sigma)$ such that $(\sigma,\hat\nu)$ is attainable by a sequence $\{y_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^{m\times n})$ in the sense that $\forall g\in C(\bar\Omega)\ \forall v_0\in\mathcal{R}$:

$$\lim_{k\to\infty}\int_\Omega g(x)v(y_k(x))\mathrm{d}x = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} g(x)v_0(s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x), \tag{1.19}$$

where

$$v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n}) := \{v_0(1+|\cdot|^p);\ v_0\in\mathcal{R}\}.$$

In particular, putting $v_0=1\in\mathcal{R}$ in (1.19) we can see that

$$\lim_{k\to\infty}(1+|y_k|^p) \ = \ \sigma \qquad \text{weakly* in } \ \mathrm{rca}(\bar\Omega). \tag{1.20}$$

If (1.19) holds, we say that $\{y_k\}_{\in\mathbb{N}}$ generates $(\sigma,\hat\nu)$. Let us denote by $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ the set of all pairs $(\sigma,\hat\nu)\in\mathrm{rca}(\bar\Omega)\times L^\infty_{\mathrm{w}}(\bar\Omega,\sigma;\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ attainable by sequences from $L^p(\Omega;\mathbb{R}^{m\times n})$; note that, taking $v_0=1$

in (1.19), one can see that these sequences must be inevitably bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$. We also denote by $\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ measures from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ generated by a sequence of gradients of some bounded sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$. The explicit description of the elements from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$, called DiPerna–Majda measures, for unconstrained sequences was given in [22], Theorem 2. In fact, it is easy to see that (1.19) can be also written in the form

$$\lim_{k\to\infty}\int_\Omega h(x,y_k(x))\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} h_0(x,s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x), \tag{1.21}$$

where $h(x,s):=h_0(x,s)(1+|s|^p)$ and $h_0\in C(\bar\Omega\otimes\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$.

We say that $\{y_k\}$ generates $(\sigma,\hat\nu)$ if (1.19) holds. Moreover, we denote $d_\sigma\in L^1(\Omega)$ the absolutely continuous (with respect to the Lebesgue measure) part of σ in the Lebesgue decomposition of σ.

Let us recall that for any $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ there is precisely one $(\sigma^\circ,\hat\nu^\circ)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\int_\Omega\int_{\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)=\int_{\bar\Omega}\int_{\mathbb{R}^{m\times n}} v_0(s)\hat\nu^\circ_x(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x) \tag{1.22}$$

for any $v_0\in C_0(\mathbb{R}^{m\times n})$ and any $g\in C(\bar\Omega)$ and $(\sigma^\circ,\hat\nu^\circ)$ is attainable by a sequence $\{y_k\}_{k\in\mathbb{N}}$ such that the set $\{|y_k|^p;\ k\in\mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$; see [22,30] for details. We call $(\sigma^\circ,\hat\nu^\circ)$ the nonconcentrating modification of $(\sigma,\hat\nu)$. We call $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ nonconcentrating if

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x)=0.$$

There is a one-to-one correspondence between nonconcentrating DiPerna–Majda measures and Young measures; *cf.* [30].

We wish to emphasize the following fact: if $\{y_k\}\in L^p(\Omega;\mathbb{R}^{m\times n})$ generates $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ and σ is absolutely continuous with respect to the Lebesgue measure it generally `does not` mean that $\{|y_k|^p\}$ is weakly relatively compact in $L^1(\Omega)$. A simple examples can be found *e.g.* in [23,30].

Having a sequence bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ generating a DiPerna−Majda measure $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ it also generates an L^p-Young measure $\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$. It easily follows from [30], Theorem 3.2.13, that

$$\nu_x(\mathrm{d}s)=d_{\sigma^\circ}(x)\frac{\hat\nu^\circ_x(\mathrm{d}s)}{1+|s|^p}\quad\text{for a.a. } x\in\Omega. \tag{1.23}$$

Note that (1.23) is well-defined as $\hat\nu^\circ_x$ is supported on $\mathbb{R}^{m\times n}$. As pointed out (in [22], Rem. 2) for almost all $x\in\Omega$

$$d_\sigma(x)=\left(\int_{\mathbb{R}^{m\times n}}\frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\right)^{-1}. \tag{1.24}$$

In fact, that (1.22) can be even improved to

$$\int_\Omega\int_{\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)=\int_{\bar\Omega}\int_{\mathbb{R}^{m\times n}} v_0(s)\hat\nu^\circ_x(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x) \tag{1.25}$$

for any $v_0\in\mathcal{R}$ and any $g\in C(\bar\Omega)$. The one-to-one correspondence between Young and DiPerna−Majda measures, in particular (see (1.23) and (1.25))

$$\int_{\mathbb{R}^{m\times n}} v(s)\nu_x(ds)=d_\sigma(x)\int_{\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(ds)$$

whenever $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$, finally yields that $\forall g\in C(\bar\Omega)$ $\forall v\in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$:

$$\lim_{k\to\infty}\int_\Omega g(x)v(y_k(x))\mathrm{d}x = \int_\Omega\int_{\mathbb{R}^{m\times n}} v(s)g(x)\nu_x(\mathrm{d}s)\,\mathrm{d}x + \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x), \tag{1.26}$$

where $\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ and $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ are Young and DiPerna–Majda measures generated by $\{y_k\}_{k\in\mathbb{N}}$, respectively. We will denote elements from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ which are generated by $\{\nabla u_k\}_{k\in\mathbb{N}}$ for some bounded $\{u_k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ by $\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$.

We will also use the following result, whose proof can be found in several places in various contexts (see [22], Lem. 1, Th. 1,2, [30], Prop. 3.2.17, or for a compactification of $\mathbb{R}^{m\times n}$ by the sphere see [1], Prop. 4.1, part (iii)).

Lemma 1.11. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open domain such that $|\partial\Omega|=0$, $\mathcal{R}$ be a separable complete subring of the ring of all continuous bounded functions on $\mathbb{R}^{m\times n}$ and $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. Then for σ_s- almost all $x\in\bar\Omega$ we have*

$$\hat\nu_x(\mathbb{R}^{m\times n})=0. \tag{1.27}$$

The following two theorems were proved in [15].

Theorem 1.12. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $1<p<+\infty$ and $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. Then then there is $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and a bounded sequence $\{u_k-u\}_{k\in\mathbb{N}}\subset W^{1,p}_0(\Omega;\mathbb{R}^m)$ such that $\{\nabla u_k\}_{k\in\mathbb{N}}$ generates $(\sigma,\hat\nu)$ if and only if the following three conditions hold*

$$\text{for a.a. } x\in\Omega\text{: } \nabla u(x) = d_\sigma(x)\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}\frac{s}{1+|s|^p}\hat\nu_x(\mathrm{d}s), \tag{1.28}$$

for almost all $x\in\Omega$ and for all $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ the following inequality is fulfilled

$$Qv(\nabla u(x))\le d_\sigma(x)\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s), \tag{1.29}$$

for σ-almost all $x\in\bar\Omega$ and all $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ with $Qv>-\infty$ it holds that

$$0\le\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s). \tag{1.30}$$

The next theorem addresses DiPerna−Majda measures generated by gradients of maps with possibly different traces.

Theorem 1.13. *Let Ω be an arbitrary bounded domain, $1<p<+\infty$ and $(\sigma,\hat\nu)\in\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ be generated by $\{\nabla u_k\}_{k\in\mathbb{N}}$ such that $w\text{-}\lim_{k\to\infty}u_k=u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. Then the conditions (1.28), (1.29) hold, and (1.30) is satisfied for σ-a.a. $x\in\Omega$.*

Remark 1.14. (i) It can happen that under the assumptions of Theorem 1.13 formula (1.30) does not hold on $\partial\Omega$. See an example in [4] showing the violation of weak sequential continuity of $W^{1,2}(\Omega;\mathbb{R}^2)\to L^1(\Omega): u\mapsto \det\nabla u$ if $\Omega=(-1,1)^2$.
(ii) In terms of Young measures, conditions (1.28) and (1.29) read, respectively: there is $u\in W^{1,p}(\Omega;\mathbb{R}^m)$:

$$\nabla u(x)=\int_{\mathbb{R}^{m\times n}} s\nu_x(\mathrm{d}s), \tag{1.31}$$

for all $v : \mathbb{R}^{m\times n} \to \mathbb{R}$, $|v| \le C(1 + |\cdot|^p)$:

$$Qv(\nabla u(x)) \le \int_{\mathbb{R}^{m\times n}} v(s)\nu_x(\mathrm{d}s). \tag{1.32}$$

Finally, we have the following result from [15].

Theorem 1.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $0 \le g \in C(\bar{\Omega})$, $v \in C(\mathbb{R}^{m\times n})$, $|v| \le C(1+|\cdot|^p)$, $C > 0$, quasiconvex, and $1 < p < +\infty$. Then the functional $I : W^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ defined as*

$$I(u) := \int_\Omega g(x)v(\nabla u(x))\,\mathrm{d}x \tag{1.33}$$

is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^m)$ if and only if for any bounded sequence $\{w_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\nabla w_k \to 0$ in measure we have $\liminf_{k\to\infty} I(w_k) \ge I(0)$.

1.4.1. Compactification of $\mathbb{R}^{m\times n}$ by the sphere

In what follows we will work mostly with a particular compactification of $\mathbb{R}^{m\times n}$, namely, with the compactification by the sphere. We will consider the following ring of continuous bounded functions

$$\mathcal{S} := \Big\{ v_0 \in C(\mathbb{R}^{m\times n}) :\ \text{there exist } c \in \mathbb{R}^{m\times n},\ v_{0,0} \in C_0(\mathbb{R}^{m\times n}),\ \text{and } v_{0,1} \in C(S^{(m\times n)-1}) \text{ s.t.}$$
$$v_0(s) = c + v_{0,0}(s) + v_{0,1}\left(\frac{s}{|s|}\right)\frac{|s|^p}{1+|s|^p} \text{ if } s \ne 0 \text{ and } v_0(0) = v_{0,0}(0)\Big\}, \tag{1.34}$$

where $S^{m\times n-1}$ denotes the $(mn-1)$-dimensional unit sphere in $\mathbb{R}^{m\times n}$. Then $\beta_{\mathcal{S}}\mathbb{R}^{m\times n}$ is homeomorphic to the unit ball $\overline{B(0,1)} \subset \mathbb{R}^{m\times n}$ *via* the mapping $d : \mathbb{R}^{m\times n} \to B(0,1)$, $d(s) := s/(1+|s|)$ for all $s \in \mathbb{R}^{m\times n}$. Note that $d(\mathbb{R}^{m\times n})$ is dense in $\overline{B(0,1)}$.

For any $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ there exists a continuous and positively p-homogeneous function $v_\infty : \mathbb{R}^{m\times n} \to \mathbb{R}$ (*i.e.* $v_\infty(\alpha s) = \alpha^p v_\infty(s)$ for all $\alpha \ge 0$ and $s \in \mathbb{R}^m$) such that

$$\lim_{|s|\to\infty} \frac{v(s) - v_\infty(s)}{|s|^p} = 0. \tag{1.35}$$

Indeed, if v_0 is as in (1.34) and $v = v_0(1 + |\cdot|^p)$ then set

$$v_\infty(s) := \left(c + v_{0,1}\left(\frac{s}{|s|}\right)\right)|s|^p \text{ for } s \in \mathbb{R}^{m\times n} \setminus \{0\}.$$

By continuity we define $v_\infty(0) := 0$. It is easy to see that v_∞ satisfies (1.35). Such v_∞ is called the *recession function* of v. It will be useful to denote

$$v_{\mathcal{S}}(s) := (c + v_{0,1})\left(\frac{s}{|s|}\right)\cdot \tag{1.36}$$

The following lemma can be found in [11, 12].

Lemma 1.16. *Let $v \in C(\mathbb{R}^{m\times n})$ be Lipschitz continuous on the unit sphere $S^{m\times n-1}$ and p-homogeneous, $p \ge 1$. Then v is p-Lipschitz,* i.e., *there is a constant $\alpha > 0$ such that for any $s_1, s_2 \in \mathbb{R}^{m\times n}$ it holds*

$$|v(s_1) - v(s_2)| \le \alpha(|s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|. \tag{1.37}$$

Remark 1.17. Notice that $\mathcal{S}$ contains all functions $v_0 := v_{0,0} + v_\infty/(1+|\cdot|^p)$ where $v_{0,0} \in C_0(\mathbb{R}^{m\times n})$ and $v_\infty : \mathbb{R}^{m\times n} \to \mathbb{R}$ is continuous and positively p-homogeneous. A weaker version of Theorem 1.13 tailored to the sphere compactification was also given in [12].

If $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$, $p > 1$, is such that $\{\nabla u_k\}$ generates $(\sigma,\hat\nu) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^{m\times n})$, $\{z_k\}$ is as in Lemma 1.10, $w_k := u_k - z_k$ for all k, and $v \in C(\mathbb{R}^{m\times n})$ is positively p-homogeneous then it follows from Lemma 1.16 (and the Stone–Weierstrass theorem on approximations of continuous functions by Lipschitz ones on a compact set) that for all $g \in C(\bar\Omega)$

$$\lim_{k\to\infty} \int_\Omega v(\nabla w_k(x)) g(x)\,\mathrm{d}x = \int_{\bar\Omega} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} g(x)\hat\nu_x(\mathrm{d}s)\,\sigma(\mathrm{d}x). \tag{1.38}$$

Indeed, let $\{\nabla u_k\}$ generate $(\sigma,\hat\nu)$ and let $\{z_k\}$ be the sequence constructed in Lemma 1.10. Denoting $w_k = u_k - z_k$ for any $k \in \mathbb{N}$ we set $R_k = \{x \in \Omega;\ \nabla w_k(x) \neq 0\}$. Lemma 1.10 asserts that $|R_k| \to 0$ as $k \to \infty$. We get from Lemma 1.1 that for any $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ p-homogeneous and any $g \in C(\bar\Omega)$

$$\begin{aligned}
&\left|\int_\Omega g(x) v(\nabla w_k(x))\,\mathrm{d}x - \int_\Omega g(x)(v(\nabla u_k(x)) - v(\nabla z_k(x)))\,\mathrm{d}x\right| \\
&\le \|g\|_{C(\bar\Omega)} \left(\int_{R_k} |v(\nabla u_k(x) - \nabla z_k(x)) - v(\nabla u_k(x))|\,\mathrm{d}x + \int_{R_k} |v(\nabla z_k(x))|\,\mathrm{d}x\right) \\
&\le C\|g\|_{C(\bar\Omega)} \int_{R_k} \left[(1 + |\nabla u_k(x) - \nabla z_k(x)|^{p-1} + |\nabla u_k|^{p-1})|\nabla z_k(x)| + (1+|\nabla z_k|^p)\right]\,\mathrm{d}x \\
&\le C'\left(\left(\int_{R_k} |\nabla z_k(x)|^p\,\mathrm{d}x\right)^{1/p} + \int_{R_k} 1 + |\nabla z_k(x)|^p\,\mathrm{d}x + \int_{R_k} |\nabla z_k(x)|\,\mathrm{d}x\right)
\end{aligned} \tag{1.39}$$

for constants $C, C' > 0$ (which may depend also on $\sup_k \|\nabla u_k\|_{L^p(\Omega)}$ and $\sup_k \|\nabla z_k\|_{L^p(\Omega)}$). The last term goes to zero as $k \to \infty$ because $\{|\nabla z_k|^p\}$ is relatively weakly compact in $L^1(\Omega)$ and $|R_k| \to 0$ as $k \to \infty$. This calculation shows that for $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ we can separate oscillation and concentration effects of $\{\nabla u_k\}$. Indeed, due to (1.26) we have for any $g \in C(\bar\Omega)$ and any $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ with $v(0) = 0$ that

$$\begin{aligned}
\lim_{k\to\infty} \int_\Omega v(\nabla w_k(x)) g(x)\,\mathrm{d}x &= \int_{\bar\Omega} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s) g(x)\,\sigma(\mathrm{d}x) \\
&= \int_{\bar\Omega} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v_\infty(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s) g(x)\,\sigma(\mathrm{d}x).
\end{aligned} \tag{1.40}$$

2. Main results

Our main result is the following explicit characterization of DiPerna–Majda measures from $\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^{m\times n})$ which are generated by gradients.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth (at least C^1) bounded domain, $1 < p < +\infty$, and $(\sigma,\hat\nu) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^{m\times n})$. Then then there is a bounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\{\nabla u_k\}_{k\in\mathbb{N}}$ generates $(\sigma,\hat\nu)$ if and only if the following three conditions hold*

$$\textit{for a.a. } x \in \Omega\textit{: } \nabla u(x) = d_\sigma(x) \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p}\hat\nu_x(\mathrm{d}s), \tag{2.1}$$

for almost all $x \in \Omega$ and for all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ the following inequality is fulfilled

$$Qv(\nabla u(x)) \le d_\sigma(x) \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s), \tag{2.2}$$

for σ-almost all $x \in \Omega$ and all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ with $Qv_\infty > -\infty$ it holds that

$$0 \le \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s), \tag{2.3}$$

and for σ-almost all $x \in \partial\Omega$ with the outer unit normal to the boundary $\varrho(x)$ and all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ with $Q_{b,\varrho(x)}v_\infty(0) = 0$ it holds that

$$0 \le \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s). \tag{2.4}$$

The following results show that sequential weak lower semicontinuity of I from (1.33) puts serious restrictions on v.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $1 < p < +\infty$. Let $0 \le g \in C(\bar{\Omega})$, $0 < g$ on $\partial\Omega$, $v \in C(\mathbb{R}^{m\times n})$, and $|v| \le C(1+|\cdot|^p)$, $C > 0$, quasiconvex such that there is a positively p-homogeneous function $v_\infty : \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfying $\lim_{|s|\to\infty}(v(s) - v_\infty(s))/|s|^p = 0$. Then the functional I defined by (1.33) is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^m)$ if and only if $Q_{b,\varrho}v_\infty(0) = 0$ for every ϱ a unit outer normal to $\partial\Omega$.*

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $1 < p < +\infty$. Let $0 \le g \in C(\bar{\Omega})$, $0 < g$ on $\partial\Omega$, $v \in C(\mathbb{R}^{m\times n})$, and $|v| \le C(1+|\cdot|^p)$, $C > 0$, quasiconvex such that there is a positively p-homogeneous function $v_\infty : \mathbb{R}^{m\times n} \to \mathbb{R}$ satisfying $\lim_{|s|\to\infty}(v(s) - v_\infty(s))/|s|^p = 0$. Let $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ weakly converge to $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. Let $|\nabla u_k|^p \to \sigma$ weakly* in $\mathrm{rca}(\bar{\Omega})$. Then the functional I defined by (1.33) satisfies $I(u) \le \liminf_{k\to\infty} I(u_k)$ if $Q_{b,\varrho(x)}v_\infty(0) = 0$ for every $\varrho(x)$, a unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$, for σ-a.a. $x \in \partial\Omega$.*

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. Let $\{u_k\} \subset W^{1,2}(\Omega;\mathbb{R}^3)$ be such that $u_k \to u$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$. Let $h(x,s) = \mathrm{Cof}\, s \cdot (a(x) \otimes \varrho(x))$, where $a, \varrho \in C(\bar{\Omega};\mathbb{R}^3)$, ϱ coincides at $\partial\Omega$ with the outer unit normal to $\partial\Omega$. Then for all $g \in C(\bar{\Omega})$*

$$\lim_{k\to\infty}\int_\Omega g(x)h(x,\nabla u_k(x))\,\mathrm{d}x = \int_\Omega g(x)h(x,\nabla u(x))\,\mathrm{d}x. \tag{2.5}$$

If, moreover, for all $k \in \mathbb{N}$ $h(\cdot,\nabla u_k) \ge 0$ almost everywhere in Ω then $h(\cdot,\nabla u_k) \to h(\cdot,\nabla u)$ weakly in $L^1(\Omega)$.

3. Necessary conditions

In this section, we show that conditions (2.1)–(2.4) are necessary. In fact, only (2.4) needs to be proved because the other conditions are stated in Theorem 1.13 which appeared in [15].

We start with the lemma proved in [15].

Lemma 3.1. *Let $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ and an open domain $\omega \subseteq \Omega$ be such that $\sigma(\partial\omega) = 0$. Let $\{y_k\}_{k\in\mathbb{N}}$ generate $(\sigma,\hat{\nu})$ in the sense (1.19). Then for all $v_0 \in \mathcal{R}$ and all $g \in C(\bar{\Omega})$*

$$\lim_{k\to\infty}\int_\Omega v(y_k)g(x)\chi_\omega(x)\,\mathrm{d}x = \int_\Omega\int_{\beta_{\mathcal{R}}\mathbb{R}^m} v_0(s)\hat{\nu}_x(\mathrm{d}s)g(x)\chi_\omega(x)\,\sigma(\mathrm{d}x), \tag{3.1}$$

where χ_ω is the characteristic function of ω in Ω.

Proposition 3.2. *Let $p > 1$ and let $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^{m\times n})$ be generated by $\{\nabla u_k\}_k$ where $\{u_k\}_k \in W^{1,p}(\Omega;\mathbb{R}^m)$ is bounded. Then for σ-almost all $x \in \partial\Omega$ it holds that for all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ with $Q_{b,\varrho(x)}v_\infty(0) = 0$*

$$0 \le \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s). \tag{3.2}$$

Proof. Let $\{\nabla u_k\}$ generates $(\sigma,\hat\nu)$, $\{u_k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$. We decompose $u_k:=z_k+w_k$ by means of Lemma 1.10. Then $\nabla w_k\to 0$ in measure and $\{\nabla w_k\}$ carries all the concentrations but no oscillations; *cf.* [12]. In particular, a simple calculation using (1.37) shows that

$$\lim_{k\to\infty}\int_\Omega v_\infty(\nabla w_k(x))\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v_\infty(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x).$$

Take $x_0\in\partial\Omega$, $\delta>0$ small enough and such that $\sigma(\partial B(x_0,\delta)\cap\bar\Omega)=0$. As $Q_{b,\varrho(x_0)}v_\infty(0)=0$ we also have that $Q_{b,\varrho(x_0)}(v_\infty+\varepsilon(|\cdot|+f(\varepsilon)|\cdot|^p))(0)\ge 0$ for $\varepsilon, f(\varepsilon)>0$. Using Lemmas 1.8 and 3.1, we have

$$\begin{aligned}0&\le\lim_{k\to\infty}\int_{B(x_0,\delta)\cap\Omega}v_\infty(\nabla w_k(x))+\varepsilon(|\nabla w_k(x)|+f(\varepsilon)|\nabla w_k(x)|^p)\,\mathrm{d}x\\&=\int_{\overline{B(x_0,\delta)\cap\Omega}}\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v_\infty(s)+\varepsilon(|s|+f(\varepsilon)|s|^p)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x).\end{aligned}\tag{3.3}$$

Sending $\varepsilon,\delta\to 0$ and using the Lebesgue−Besicovitch theorem [9] we get that for σ-almost all $x_0\in\partial\Omega$ it holds that

$$0\le\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v_\infty(s)}{1+|s|^p}\hat\nu_{x_0}(\mathrm{d}s)=\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_{x_0}(\mathrm{d}s).\tag{3.4}$$

We continue similarly as in [12]. The previous calculation yields the existence of a σ-null set $E_v\subset\partial\Omega$ such that

$$0\le\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)$$

if $x\notin E_v$ and $\varrho(x)=\varrho(x_0)=:\rho$. Let $\{v_0^k\}_{k\in\mathbb{N}}$ be a dense subset of $\mathcal S$, so that $\{v^k\}_{k\in\mathbb{N}}=\{v_0^k(1+|\cdot|^p)\}_{k\in\mathbb{N}}\subset\Upsilon^p_{\mathcal S}(\mathbb{R}^{m\times n})$. We define

$$E=\bigcup_k\ \bigcup_{\{j\in\mathbb{N};\ Q_{b,\rho}(v^k_\infty+(1/j)(1+|\cdot|^p)(0)>-\infty\}}E_{v^k_\infty+(1/j)(1+|\cdot|^p)}.$$

Clearly $\sigma(E)=0$. Fix $x\in(\Omega\setminus E)$, $v\in\Upsilon^p_{\mathcal S}(\mathbb{R}^{m\times n})$ such that $Q_{b,\rho}v_\infty(0)>-\infty$ and choose a subsequence (not relabeled) $\{v_0^k\}_{k\in\mathbb{N}}$ such that

$$v_0^k\to v_0\text{ in }C(\beta_{\mathcal S}\mathbb{R}^{m\times n})\text{ and }\|v_0^k-v_0\|_{C(\beta_{\mathcal S}\mathbb{R}^{m\times n})}<\frac{1}{j(k)},$$

where $j(k)\to\infty$ if $k\to\infty$. We have

$$\begin{aligned}v^k(s)+\frac{1}{j(k)}(1+|s|^p)&\ge v^k(s)+(1+|s|^p)\|v_0^k-v_0\|_{C(\beta_{\mathcal S}\mathbb{R}^{m\times n})}\\&\ge v^k(s)+|v_0^k(s)-v_0(s)|(1+|s|^p)\ge v(s).\end{aligned}$$

Thus, $Q_{b,\rho}(v^k_\infty+\frac{1}{j(k)}(1+|s|^p))+>-\infty$, as well, and because $x\notin E$ then $x\notin E_{v^k_\infty+(1/j(k))(1+|\cdot|^p)}$ and

$$\begin{aligned}0&\le\lim_{k\to\infty}\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\left(v_0^k(s)+\frac{1}{j(k)}\right)\hat\nu_x(\mathrm{d}s)=\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(\mathrm{d}s)\\&=\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s).\end{aligned}$$

□

4. Sufficient conditions

The goal of this section is to show that conditions (2.1)–(2.4) are sufficient, as well. We will use the following lemma from [12] concerning Young measures from $\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ which are generated by sequences of gradients.

If $B(0,1)$ is the open unit ball in $\mathbb{R}^n$ centered at zero and $\varrho\in\mathbb{R}^n$ a unit vector we denote

$$B_\varrho=\{x\in\mathbb{R}^n;\ x\in B(0,1)\cap\{x\in\mathbb{R}^n;\ \varrho\cdot x<0\}\}$$

and $\partial B_\varrho\supset\Gamma_\varrho=\{x\in\partial B_\varrho;\ \varrho\cdot x=0\}$.

We define two sets of measures:

$$A^\varrho:=\{\mu\in\mathrm{rca}(\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n});\ \mu\geq 0,\ \langle\mu;v_0\rangle\geq 0 \text{ for } v_0\in\mathcal{S} \text{ if } Q_{b,\varrho}v_\infty(0)=0\}$$

and

$$H^\varrho:=\{\bar\delta_{\varrho,\nabla u};\ u\in W^{1,p}_0(B(0,1);\mathbb{R}^m)\},$$

where for all $v\in$ and positively p-homogeneous

$$\left\langle\bar\delta_{\varrho,\nabla u},v_0\right\rangle=|B_\varrho|^{-1}\int_{B_\varrho}v_{\mathcal{S}}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)|\nabla u(x)|^p\,\mathrm{d}x.$$

As $\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}\cong S^{m\times n-1}$, the unit sphere in $\mathbb{R}^{m\times n}$ centered at zero we consider both H^ϱ and A^ϱ as sets of measures on the unit sphere. Moreover, $H^\varrho\subset A^\varrho$. Notice, that by the definition of $v_{\mathcal{S}}$ we have

$$\int_{B_\varrho}v_{\mathcal{S}}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)|\nabla u(x)|^p\,\mathrm{d}x=\int_{B_\varrho}v_\infty(\nabla u(x))\,\mathrm{d}x.$$

Moreover, in view of Remark 1.7 it is sufficient to consider only $u\in W^{1,p}_0(B(0,1);\mathbb{R}^m)\}$ which are symmetric with respect to the plane $\{x\in\mathbb{R}^n;\ \varrho\cdot x=0\}$ in the definition of H^ϱ.

Lemma 4.1. *Let $n\geq 2$. Then the set H^ϱ is convex.*

Proof. Consider $u_1,u_2\in W^{1,p}_0(B(0,1);\mathbb{R}^m)\}$. We need to show that for any $0\leq\lambda\leq 1$ $\lambda\bar\delta_{\varrho,\nabla u_1}+(1-\lambda)\bar\delta_{\varrho,\nabla u_2}\in H^\varrho$. Take $x_0\in B(0,1)\cap\{x\in\mathbb{R}^n;\ \varrho\cdot x=0\}$ such that $|x_0|=1/2$. Define $\tilde u_1(x):=5^{n/p-1}u_1(5x)$ and $\tilde u_2(x):=5^{n/p-1}u_1(5(x-x_0))$. We see that $\tilde u_1\in W^{1,p}_0(B(0,1/5);\mathbb{R}^m)$ and $\tilde u_1\in W^{1,p}_0(B(x_0,1/5);\mathbb{R}^m)$, and we may extend them by zero to the whole $\mathbb{R}^n$ (we denote the extension again $\tilde u_1$ and $\tilde u_2$), so that in particular, $\tilde u_1,\tilde u_2\in W^{1,p}_0(B(0,1);\mathbb{R}^m)$ and they have disjoint supports. Take $u:=\lambda^{1/p}\tilde u_1+(1-\lambda)^{1/p}\tilde u_2(x)$. Then

$$\begin{aligned}\int_{B_\varrho}v(\nabla u(x))\,\mathrm{d}x&=\lambda\int_{B_\varrho}v(5^{n/p}\nabla u_1(5x))\,\mathrm{d}x+(1-\lambda)\int_{B_\varrho\cap(B(x_0,1/5)}v(5^{n/p}\nabla u_1(5(x-x_0))\,\mathrm{d}x\\&=\lambda\int_{B_\varrho}v(\nabla u_1(y))\,\mathrm{d}y+(1-\lambda)\int_{B_\varrho}v(\nabla u_2(y))\,\mathrm{d}y.\end{aligned}$$

This proves the claim. □

Remark 4.2. The case $n=1$ is easy because then quasiconvexity at the boundary reduces to convexity and convex functions are bounded from below by an affine function. Hence, (2.3) and (2.4) always hold.

Proposition 4.3. *The set A^ϱ is the weak* closure of H^ϱ.*

Proof. It is a standard application of the Hahn−Banach theorem. Clearly, $H^\varrho\subset A^\varrho$. Take $v_0\in\mathcal{S}$ such that $\langle\mu,v_0\rangle\geq a$ for some $a\in\mathbb{R}$ and for all $\mu\in H^\varrho$. Then also

$$\inf_{u\in W^{1,p}_0(B(0,1);\mathbb{R}^m)}\int_{B_\varrho}v_\infty(\nabla u(x))\,\mathrm{d}x\geq a$$

and by p-homogeneity we have $\inf_{u\in W^{1,p}_0(B(0,1);\mathbb{R}^m)}\int_{B_\varrho}v_\infty(\nabla u(x))\,\mathrm{d}x=0$. Therefore $0\geq a$ and $Q_{b,\varrho}v_\infty(0)=0$. By the definition of A^ϱ we see that $\langle\pi,v\rangle\geq 0\geq a$ for all $\pi\in A^\varrho$. □

The following two sets of measures were defined in [12]

$$A := \{\mu \in \mathrm{rca}(\beta_{\mathcal{S}}\mathbb{R}^{m\times n} \setminus \mathbb{R}^{m\times n});\ \mu \geq 0, \langle \mu; v_0\rangle \geq 0 \text{ for } v_0 \in \mathcal{S} \text{ if } Qv_\infty(0) = 0\}$$

and

$$H := \{\bar{\delta}_{\nabla u};\ u \in W_0^{1,p}(B(0,1);\mathbb{R}^m)\},$$

where for all $v \in$ and positively p-homogeneous

$$\langle \bar{\delta}_{\nabla u}, v_0 \rangle = |B(0,1)|^{-1} \int_{B(0,1)} v_{\mathcal{S}}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right) |\nabla u(x)|^p \,\mathrm{d}x.$$

We have the following proposition.

Proposition 4.4 (see [12], Prop. 6.1)**.** *The set A is the weak* closure of H.*

The following theorem formulates sufficient conditions for $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^{m\times n})$ to be generated by gradients.

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $1 < p < +\infty$, and $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^{m\times n})$. Then then there is a bounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\{\nabla u_k\}_{k\in\mathbb{N}}$ generates $(\sigma, \hat{\nu})$ if the following three conditions hold*

$$\text{for a.a. } x \in \Omega\text{: } \nabla u(x) = d_\sigma(x) \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s), \tag{4.1}$$

for almost all $x \in \Omega$ and for all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ the following inequality is fulfilled

$$Qv(\nabla u(x)) \leq d_\sigma(x) \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s), \tag{4.2}$$

for σ-almost all $x \in \Omega$ and all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ with $Qv_\infty > -\infty$ it holds that

$$0 \leq \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s), \tag{4.3}$$

and for σ-almost all $x \in \partial\Omega$ with the outer unit normal to the boundary $\varrho(x)$ and all $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^{m\times n})$ with $Q_{b,\varrho(x)}v_\infty(0) = 0$ it holds that

$$0 \leq \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s). \tag{4.4}$$

Before we give the proof we just define a restriction of $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ to a set $\bar{\omega} \subset \Omega$. Naturally, this is a couple $(\pi, \hat{\gamma}) \in \mathcal{DM}^p_{\mathcal{R}}(\omega;\mathbb{R}^m)$ such that $\hat{\gamma}_x = \hat{\nu}_x$ if $x \in \bar{\omega}$ and π is the restriction of σ to $\bar{\omega}$.

Proof. By the assumptions of the theorem the corresponding Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ given by (1.23) is generated by gradients of mappings from $W^{1,p}(\Omega;\mathbb{R}^m)$. By Lemma 1.10 we suppose that this Young measure is generated by $\{\nabla z_k\}_{k\in\mathbb{N}}$ such that $\{|\nabla z_k|^p\}$ is weakly converging in $L^1(\Omega;\mathbb{R}^m)$ and $\{z_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$. Then we look for $\{w_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $\{\nabla w_k\}$ generates given concentrations but no oscillations. Then $\{\nabla z_k + \nabla w_k\}$ generates the whole DiPerna−Majda measure, see (1.40). If $\sigma(\partial\Omega) = 0$ the proof is exactly the same as in [12], page 753. By Lemma 1.10 sought $\{w_k\}$ is such that (i) $w_k \to 0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$ and (ii) $\nabla w_k \to 0$ in measure. Therefore, it is sufficient to find a sequence of gradients whose Young measure is $\{\delta_0\}_{x\in\Omega}$ and whose DiPerna−Majda satisfies (4.3), (4.4), and (4.1) and (4.2) hold with $u = 0$. The proof is divided into two steps. The first step deals with the situation that σ only concentrates on the boundary. Two cases are considered – (a) the singular part of σ is a weighted sum of Dirac masses and – (b) the general case. The second step assumes that σ is arbitrary.

(i) Suppose first that σ concentrates only at the boundary of Ω.

Notice that from (4.4) it follows that for σ-almost all $x \in \partial\Omega$ $\hat{\nu}_x \in A^{\varrho(x)}$. Hence, there is a bounded sequence $\{u_k\} \subset W^{1,p}(B(0,1);\mathbb{R}^m)$, each u_k is symmetric with respect to the plane $\{y \in \mathbb{R}^n;\ \varrho(x)\cdot y = 0\}$ and

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v_\infty(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s) = \lim_{k\to\infty} |B_{\varrho(x)}|^{-1}\int_{B_{\varrho(x)}} v_\infty(\nabla u_k(y))\,\mathrm{d}y,$$

whenever $Q_{b,\varrho(x)}v_\infty(0) = 0$. By symmetry, there is $\hat{\mu}_x \in \mathrm{rca}(\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n})$ such that for the same v it holds that

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v_\infty(s)}{1+|s|^p}\hat{\mu}_x(\mathrm{d}s) = \lim_{k\to\infty} |B_{\varrho(x)}|^{-1}\int_{B(0,1)\setminus B_{\varrho(x)}} v_\infty(\nabla u_k(y))\,\mathrm{d}y.$$

Thus,

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\mu}_x(\mathrm{d}s) = \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0\left(s\left(\mathbb{I}-2\varrho(x)\otimes\varrho(x)\right)\right)\hat{\nu}_x(\mathrm{d}s) \tag{4.5}$$

for all $v_0 \in \mathcal{S}$. Altogether, there is a bounded open $O \subset \mathbb{R}^n$, $\Omega \subset O$ such that $(\pi,\hat{\gamma}) \in \mathcal{DM}^p_{\mathcal{S}}(O;\mathbb{R}^n)$ with

$$\hat{\gamma}_x := \begin{cases} \frac{1}{2}\hat{\nu}_x + \frac{1}{2}\hat{\mu}_x \text{ if } x \in \partial\Omega, \\ \delta_0 \text{ if } x \in \bar{O}\setminus\partial\Omega, \end{cases} \tag{4.6}$$

and

$$\pi := \begin{cases} 2\sigma \text{ in } \partial\Omega, \\ \mathcal{L}^n \text{ in } \bar{O}\setminus\partial\Omega. \end{cases} \tag{4.7}$$

This means, in particular, that $\hat{\gamma}_x \in A$ defined in Lemma 4.4 and that π is the n-dimensional Lebesgue measure in $\bar{O}\setminus\partial\Omega$. Moreover, π does not concentrate on ∂O and by Theorem 1.12, see also [12], Theorem 1.1, $(\pi,\hat{\gamma})$ is generated by gradients $\{\nabla w_k\}$ where $\{w_k\} \subset W^{1,p}(O;\mathbb{R}^m)$ is bounded.

(a) Assume first, that that the singular part of π, π_s is equal to $\sum_{i=1}^N a_i\delta x_i$. We know from Proposition 4.4 that if $Qv_\infty(0) = 0$ then

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v_\infty(s)}{1+|s|^p}\hat{\gamma}_{x_i}(\mathrm{d}s) = \lim_{k\to\infty}|B(0,1)|^{-1}\int_{B(0,1)} v_\infty(\nabla u^i_k(x))\,\mathrm{d}x$$

for $1 \le i \le N$ and $u^i_k \subset W^{1,p}_0(B(0,1);\mathbb{R}^m)$ a bounded sequence in k. In view of (4.6), (4.5), and Proposition 4.3 we see that u^i_k can be taken symmetric with respect to the plane $\{y;\ \varrho(x_i)\cdot y = 0\}$. Thus, following [12]

$$w_k(x) := k^{n/p-1}|B(0,1)|^{-1/p}\sum_{i=1}^N a_i^{1/p}u^i_k(k(x-x_i))$$

is such that $\{\nabla(z_k + w_k)\}$ generates $(\pi,\hat{\gamma})$ and its restriction to Ω generates (by symmetry) $(\sigma,\hat{\nu})$.

(b) Take $l \in \mathbb{N}$. There exists a finite partition $\mathcal{P}_l = \{O^l_j\}_{j=1}^{J(l)}$ of $\bar{O}$ such that $O^l_{j_1}\bigcap O_{j_2} = \emptyset,\ 1 \le j_1 < j_2 \le J(l)$ and all O^l_j are measurable with $\mathrm{diam}(O^l_j) < 1/l$. Besides, we may suppose that, for any $l \in \mathbb{N}$, the partition

$\mathcal{P}_{l+1}$ is a refinement of $\mathcal{P}_l$, $\mathrm{int}(O^l_j) \neq \emptyset$ for all j and that σ-almost all $x \in \partial\Omega$ belong to $\mathrm{int}(O^l_j)$ for some j. Let π_s be the singular part of π. We set $a^l_i = \pi_s(O^l_i)$, where π_s is the singular part of π. Let us put

$$N(l) = \{1 \le j \le J(l);\ a^l_j \neq 0\}.$$

If $i \in N(l)$ take $x_i \in \mathrm{int}(O^l_i)$ such that $x_i \in \partial\Omega$ if $\mathrm{int}(O^l_i) \cap \partial\Omega \neq \emptyset$. I learned the following argument from Stefan Krömer. We define for $x \in \Omega^l_i \cap \partial\Omega$ rotation matrices $R_{x_i l}(x)$ such that $\varrho(x) = R_{x_i l}(x)\varrho(x_i)$ for every $x \in \partial\Omega \cap O^l_i$, hence $R_{x_i l}(x_i) = \mathbb{I}$. Notice that if v is quasiconvex at the boundary at zero with the normal ϱ then $s \mapsto v(sR_{x_i l})$ is quasiconvex at the boundary at zero with the normal $R_{x_i l}\varrho$. Define a measure $(\pi^l, \hat{\gamma}^l)$ by the formula $\pi^l(\mathrm{d}x) = d_\pi(x) + \sum_{i \in N(l)} a^l_i \delta_{x_i}$ and

$$\hat{\gamma}^l_x = \begin{cases} \hat{\gamma}_x & \text{if } x \neq x_i \\ \hat{\gamma}^l_{x_i} & \text{if } x = x_i, \end{cases} \tag{4.8}$$

where $\mathrm{supp}\ \hat{\gamma}^l_{x_i} \subset \beta_{\mathcal{S}}\mathbb{R}^{m\times n} \setminus \mathbb{R}^{m\times n}$ and for any $v_0 \in \mathcal{S}$

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} v_0(s)\hat{\gamma}^l_{x_i}(\mathrm{d}s) = \frac{1}{\pi_s(O^l_i)} \int_{O^l_i} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} v_0(sR^{-1}_{x_i l}(x))\hat{\gamma}_x(\mathrm{d}s)\,\pi_s(\mathrm{d}x). \tag{4.9}$$

Using Lemma 1.11 we can equivalently rewrite (4.9) as

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(sR^{-1}_{x_i l})\hat{\gamma}^l_{x_i}(\mathrm{d}s) = \frac{1}{\pi_s(O^l_i)} \int_{O^l_i} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(sR^{-1}_{x_i l}(x))\hat{\gamma}_x(\mathrm{d}s)\,\pi_s(\mathrm{d}x).$$

Theorem 1.12 implies that $(\pi^l, \hat{\gamma}^l) \in \mathcal{GDM}^p_{\mathcal{S}}(O;\mathbb{R}^m)$. Indeed, the fact that $(\pi^l, \hat{\gamma}^l) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ is checked by using Proposition A.1. Moreover, an easy verification shows that (2.1)–(2.3) are also satisfied for $(\pi^l, \hat{\gamma}^l)$.

Let $\{y^l_k\}_{k\in\mathbb{N}} \subset W^{1,p}(O;\mathbb{R}^m)$ be such that $\{\nabla y^l_k\}_{k\in\mathbb{N}}$ generates $(\pi^l, \hat{\gamma}^l)$. We have for any $l \in \mathbb{N}$

$$\lim_{k\to\infty} \int_O (1 + |\nabla y^l_k(x)|^p)\,\mathrm{d}x = \pi^l(\bar{O}) = \pi(\bar{O}) \tag{4.10}$$

and for any $v_0 \in \mathcal{S}$ and any $g \in C(\bar{O})$

$$\begin{aligned}
&\lim_{l\to\infty} \left| \int_{\bar{O}} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} v_0(s)\hat{\gamma}^l_x(\mathrm{d}s)g(x)\,\pi^l(\mathrm{d}x) - \int_{\bar{O}} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} v_0(s)\hat{\gamma}_x(\mathrm{d}s)g(x)\,\pi(\mathrm{d}x) \right| \\
&= \lim_{l\to\infty} \left| \sum_{i\in N(l)} g(x_i)\pi_s(O^l_i) \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(sR^{-1}_{x_i l}(x))\hat{\gamma}^l_{x_i}(\mathrm{d}s) - \int_{\bar{O}} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\gamma}_x(\mathrm{d}s)g(x)\,\pi_s(\mathrm{d}x) \right| \\
&= \lim_{l\to\infty} \left| \sum_{i\in N(l)} \left(\int_{O^l_i} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(sR^{-1}_{x_i l}(x))\hat{\gamma}_x(\mathrm{d}s)g(x_i)\pi_s(\mathrm{d}x) - \int_{O^l_i} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} v_0(s)\hat{\gamma}_x(\mathrm{d}s)g(x)\,\pi_s(\mathrm{d}x) \right) \right| \\
&\le \lim_{l\to\infty} \sum_{i\in N(l)} \int_{O^l_i} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} |v_0(sR^{-1}_{x_i l}(x))|\hat{\gamma}_x(\mathrm{d}s)|g(x) - g(x_i)|\,\pi_s(\mathrm{d}x) \\
&\quad + \lim_{l\to\infty} \sum_{i\in N(l)} \int_{O^l_i} \int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} |v_0(sR^{-1}_{x_i l}(x)) - v_0(s)|\hat{\gamma}_x(\mathrm{d}s)|g(x)|\,\pi_s(\mathrm{d}x) \\
&\le C\pi_s(\bar{O}) \lim_{l\to\infty} \left(M_g\left(\frac{1}{l}\right) + M_{v_0}\left(\frac{\tilde{C}}{l}\right) \right) = 0,
\end{aligned}$$

where $|v_0|+|g|\le C$, $\tilde C>0$, and M_g and M_{v_0} are the moduli of continuity of the uniformly continuous $g\in C(\bar O)$ and $v_0\in C(\beta_{\mathcal S}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n})$. Here we used the fact that $x\mapsto R(x)$ is continuous for $x\in\partial\Omega$. Hence, we get for any $v\in\Upsilon^p_{\mathcal R}(\mathbb{R}^{m\times n})$ and any $g\in C(\bar O)$

$$\lim_{l\to\infty}\lim_{k\to\infty}\int_O v(\nabla y^l_k(x))g(x)\,\mathrm{d}x=\int_{\bar O}\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}} v_0(s)\hat\gamma_x(\mathrm{d}s)g(x)\,\pi(\mathrm{d}x).$$

However, we know by part (i) (a) of the proof that

$$\lim_{l\to\infty}\lim_{k\to\infty}\int_\Omega v(\nabla y^l_k(x))g(x)\,\mathrm{d}x=\int_{\bar \Omega}\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\,\sigma(\mathrm{d}x).$$

The proof of this case is finished by the diagonalization argument.

(ii) Assume that σ is arbitrary. Take $1>\varepsilon>0$ and take $\Omega(\varepsilon)\subset\Omega$ such that $\sigma(\Omega\setminus\overline{\Omega(\varepsilon)})\to0$ as $\varepsilon\to1$. Moreover, we suppose that $\sigma(\partial\Omega(\varepsilon))=0$. By Theorem 1.12 there is $\{c^\varepsilon_k\}_{k\in\mathbb N}\subset W^{1,p}_0(\Omega(\varepsilon);\mathbb{R}^m)$ so that ∇c^ε_k generates the restriction of $(\sigma,\hat\nu)$ to $\overline{\Omega(\varepsilon)}$. We can thus extend c^ε_k by zero to the whole Ω (without changing the notation).

Let us define

$$\hat\beta^\varepsilon_x:=\begin{cases}\hat\nu_x \text{ if } x\in\overline{\Omega(\varepsilon)},\\ \delta_0 \text{ if } x\in\Omega\setminus\overline{\Omega(\varepsilon)},\\ \hat\nu_x \text{ if } x\in\partial\Omega.\end{cases}\tag{4.11}$$

and

$$\zeta^\varepsilon:=\begin{cases}\sigma \text{ in } \overline{\Omega(\varepsilon)},\\ \mathcal{L}^n \text{ in } \Omega\setminus\overline{\Omega(\varepsilon)},\\ \sigma \text{ if } x\in\partial\Omega.\end{cases}\tag{4.12}$$

Then $(\zeta^\varepsilon,\hat\beta^\varepsilon)\in\mathcal{GDM}^p_{\mathcal S}(\Omega;\mathbb{R}^{m\times n})$ by the construction from (i) and Theorem 1.12. Namely, using (i) we construct a sequence of gradients $\{\nabla b^\varepsilon_k\}_k$ generating $(\zeta^\varepsilon,\hat\beta^\varepsilon)$ restricted to $\bar\Omega\setminus\overline{\Omega(\varepsilon)}$. This sequence does not concentrate on $\partial\overline{\Omega(\varepsilon)}$ and can be chosen so, that $\{b^\varepsilon_k\}_k\subset W^{1,p}_0(\Omega(\varepsilon);\mathbb{R}^m)$. Then using Theorem 1.12 we have that $\{\nabla b^\varepsilon_k+\nabla c^\varepsilon_k\}_k$ generates $(\zeta^\varepsilon,\hat\beta^\varepsilon)$. Moreover, ζ^ε is majorized by σ for every $1\ge\varepsilon>0$, therefore there is a uniform bound on $\{\nabla b^\varepsilon_k+\nabla c^\varepsilon_k\}_k$ in $L^p(\Omega;\mathbb{R}^{m\times n})$ independently of ε. We can then shift $b^\varepsilon_k+c^\varepsilon_k$ by a constant (dependent on k and ε) so that its average over Ω is zero. The Poincaré inequality then gives us a uniform bound on $\{b^\varepsilon_k+c^\varepsilon_k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$.

Finally notice that for all $v_0\in\mathcal S$, all $g\in C(\bar\Omega)$ and $\{\varepsilon^l\}_{l\in\mathbb N}\subset(0,1)$, $\lim_{l\to\infty}\varepsilon^l=1$, such that $\sigma(\partial\Omega(\varepsilon^l))=0$ it holds

$$\begin{aligned}&\lim_{l\to\infty}\left|\int_{\bar\Omega}\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)-\int_{\bar\Omega}\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}} v_0(s)\hat\beta^{\varepsilon^l}_x(\mathrm{d}s)g(x)\zeta^{\varepsilon^l}(\mathrm{d}x)\right|\\ &=\lim_{l\to\infty}\left|\int_{\Omega\setminus\overline{\Omega(\varepsilon^l)}}\int_{\beta_{\mathcal S}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)-\int_{\Omega\setminus\Omega(\varepsilon^l)} v_0(0)g(x)\,\mathrm{d}x\right|\\ &\le\lim_{l\to\infty}C(\sigma(\Omega\setminus\overline{\Omega(\varepsilon^l)})+\mathcal{L}^n(\Omega\setminus\Omega(\varepsilon^l))=0,\end{aligned}$$

where $C>0$ is a constant depending on v_0 and g. Finally, we finish the proof by a diagonalization argument as $\mathcal S$ and $C(\bar\Omega)$ are separable. The theorem is proved. □

5. Proofs of Theorems 2.2–2.4

Proof of Theorem 2.2. Take $\{u_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_k \to u$ weakly. Then multiplying the inequalities in Theorem 2.1 by g as in the theorem and integrating over $\bar{\Omega}$, we have for a subsequence realizing $\liminf_{k\to\infty} I(u_k)$ (not relabeled) and generating $(\sigma,\hat{\nu})$ that

$$\lim_{k\to\infty} I(u_k) = \lim_{k\to\infty}\int_\Omega v(\nabla u_k(x))g(x))\,\mathrm{d}x = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x) \geq I(u),$$

which finishes the proof of the "if part".

To show the "only if part" of the assertion we assume that I is sequentially weakly lower semicontinuous and want to show that $Q_{b,\varrho}v_\infty(0)=0$. Consider $x_0\in\partial\Omega$ and $u\in W^{1,p}_0(B(0,1);\mathbb{R}^m)$ and extend it by zero to the whole $\mathbb{R}^n$. Define for $x\in\mathbb{R}^n$ and $k\in\mathbb{N}$ $u_k(x)=k^{n/p-1}u(k(x-x_0))$, *i.e.*, $u_k \rightharpoonup 0$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ and assume that ϱ is the outer unit normal to $\partial\Omega$ at x_0. The Young measure generated by $\{\nabla u_k\}$ is just $\{\delta_0\}_{x\in\Omega}$ because $\nabla u_k\to 0$ in measure. Hence, we get

$$\begin{aligned}\lim_{k\to\infty} I(u_k) &= I(0) + \int_{\bar{\Omega}}\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)\\ &= I(0) + \int_{\bar{\Omega}}\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v_\infty(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)\\ &= I(0) + g(x_0)\int_{B(0,1)\cap\{x\in\mathbb{R}^n;\,\varrho\cdot x\le 0\}} v_\infty(\nabla u(x))\,\mathrm{d}x.\end{aligned}$$

If the last term is negative for some $u\in W^{1,p}_0(B(0,1);\mathbb{R}^m)$, *i.e.*, if $Q_{b,\varrho}v_\infty(0)=-\infty$ then $\lim_{k\to\infty} I(u_k)<I(0)$, so that I is not sequentially weakly lower semicontinuous, which would contradict our assumption. The assertion is proved. □

Proof of Theorem 2.3. It is just a corollary of Theorem 2.2. □

Proof of Theorem 2.4. Let $m=n=3$. Functions $s\mapsto\pm\mathrm{Cof}\ s$ are both quasiconvex [5] and, as already mentioned in [34], $s\mapsto \pm a\cdot[\mathrm{Cof}\ s]\varrho$ is quasiconvex at the boundary with the normal ϱ. Thus, all the inequalities in Theorem 2.1 are equalities if applied to $v=\pm h(x,\cdot)$ for a fixed $x\in\bar{\Omega}$. Hence, if $\{\nabla u_k\}$ generates $(\sigma,\hat{\nu})$ on the domain $\bar{\Omega}$ and $u_k\to u$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$ we have for all $g\in C(\bar{\Omega})$

$$\lim_{k\to\infty}\int_\Omega g(x)h(x,\nabla u_k(x))\,\mathrm{d}x = \int_{\bar{\Omega}}\int_{\beta_{\mathcal{S}}\mathbb{R}^{m\times n}}\frac{h(x,s)}{1+|s|^2}\hat{\nu}_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x) = \int_\Omega g(x)h(x,\nabla u(x))\,\mathrm{d}x.$$

If $0\le h(x,\nabla u_k(x))$ the result follows by Lemma A.2 in the Appendix. □

Appendix A

The following proposition from [22] explicitly characterizes elements of $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$.

Proposition A.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open domain, $\mathcal{R}$ be a separable complete subring of the ring of all continuous bounded functions on $\mathbb{R}^{m\times n}$ and $(\sigma,\hat{\nu})\in\mathrm{rca}(\bar{\Omega})\times L^\infty_{\mathrm{w}}(\bar{\Omega},\sigma;\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ and $1\le p<+\infty$. Then the following two statements are equivalent with each other:*

(i) *the pair $(\sigma,\hat{\nu})$ is the DiPerna–Majda measure, i.e. $(\sigma,\hat{\nu})\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$;*
(ii) *The following properties are satisfied simultaneously:*
   (1) *σ is positive,*

(2) $\sigma_{\hat\nu} \in \mathrm{rca}(\bar\Omega)$ *defined by* $\sigma_{\hat\nu}(\mathrm{d}x) = (\int_{\mathbb{R}^{m\times n}} \hat\nu_x(\mathrm{d}s))\sigma(\mathrm{d}x)$ *is absolutely continuous with respect to the Lebesgue measure* *(*$d_{\sigma_{\hat\nu}}$ *will denote its density),*

(3) *for a.a.* $x \in \Omega$ *it holds*

$$\int_{\mathbb{R}^{m\times n}} \hat\nu_x(\mathrm{d}s) > 0, \qquad d_{\sigma_{\hat\nu}}(x) = \left(\int_{\mathbb{R}^{m\times n}} \frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\right)^{-1} \int_{\mathbb{R}^{m\times n}} \hat\nu_x(\mathrm{d}s),$$

(4) *for* σ*-a.a.* $x \in \bar\Omega$ *it holds*

$$\hat\nu_x \geq 0, \qquad \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \hat\nu_x(\mathrm{d}s) = 1.$$

Lemma A.2. *Let* $1 \leq p < +\infty$*,* $0 \leq h_0 \in C(\bar\Omega \times \beta_{\mathcal{R}}\mathbb{R}^{m\times n})$*, and let* $\{\nabla u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{m\times n})$ *with* $u_k \in W^{1,p}(\Omega;\mathbb{R}^m)$*, generate* $(\sigma,\hat\nu) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$*. Let* $h(x,s) := h_0(x,s)(1+|s|^p)$*. Then* $\{h(x,\nabla u_k)\}_{k\in\mathbb{N}}$ *is weakly relatively compact in* $L^1(\Omega)$ *if and only if*

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} h_0(x,s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x) = 0. \tag{A.1}$$

Proof. We follow the proof of [30], Lemma 3.2.14(i). Suppose first that (A.1) holds. For $\varrho \geq 0$ define the function $\xi^\varrho : \mathbb{R}^{m\times n} \to \mathbb{R}$

$$\xi^\varrho(s) := \begin{cases} 0 & \text{if } |s| \leq \varrho, \\ |s|-\varrho & \text{if } \varrho \leq |s| \leq \varrho+1, \\ 1 & \text{if } |s| \geq \varrho+1. \end{cases}$$

Note that always $\xi^\varrho \in \mathcal{R}$, hence $\xi^\varrho h_0(x,\cdot) \in \mathcal{R}$ because $\mathcal{R}$ is closed under multiplication. We have due to the Lebesgue Dominated Convergence Theorem

$$\lim_{\varrho\to\infty}\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus B(0,\varrho)} h_0(x,s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x) = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} h_0(x,s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x) = 0.$$

Let $\varepsilon > 0$ and ϱ be large enough so that

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \xi^\varrho(s)h_0(x,s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x) \leq \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus B(0,\varrho)} h_0(x,s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x) \leq \frac{\varepsilon}{2},$$

and choose $k_\varrho \in \mathbb{N}$ such that, if $k \geq k_\varrho$, then

$$\left|\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \xi^\varrho(s)h_0(x,s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x) - \int_\Omega \xi^\varrho(\nabla u_k(x))h(x,\nabla u_k(x))\,\mathrm{d}x\right| \leq \frac{\varepsilon}{2}\cdot$$

Therefore, if $k \geq k_\varrho$ then $\int_\Omega \xi^\varrho(\nabla u_k(x))h(x,\nabla u_k(x))\,\mathrm{d}x \leq \varepsilon$, and so

$$\int_{\{x\in\Omega:\ |\nabla u_k(x)|\geq\varrho+1\}} h(x,\nabla u_k(x))\,\mathrm{d}x \leq \int_\Omega \xi^\varrho(\nabla u_k(x))h(x,\nabla u_k(x))\,\mathrm{d}x \leq \varepsilon.$$

As $0 \leq h(x,\cdot) \leq C(1+|\cdot|^p)$ for some $C > 0$, we get for $K \geq C(1+(\varrho+1)^p)$ that

$$\int_{\{x\in\Omega:\ |h(x,\nabla u_k(x))|\geq K\}} h(x,\nabla u_k(x))\,\mathrm{d}x \leq \int_{\{x\in\Omega:\ |\nabla u_k(x)|\geq\varrho+1\}} h(x,\nabla u_k(x))\,\mathrm{d}x \leq \varepsilon.$$

Clearly, the finite set $\{h(x,\nabla u_k)\}_{k=1}^{k_\varrho}$ is weakly relatively compact in $L^1(\Omega)$, which means that for $K_0 > 0$ sufficiently large and $1 \leq k \leq k_\varrho$

$$\int_{\{x\in\Omega:\ |h(x,\nabla u_k(x))|\geq K_0\}} h(x,\nabla u_k(x))\,\mathrm{d}x \leq \varepsilon.$$

Hence,

$$\sup_{k\in\mathbb{N}}\int_{\{x\in\Omega:\ |h(x,\nabla u_k(x))|\geq\max(K_0,K)\}} h(x,\nabla u_k(x))\,\mathrm{d}x\leq\varepsilon,$$

and $\{h(x,\nabla u_k)\}$ is relatively weakly compact in $L^1(\Omega)$ by the Dunford−Pettis criterion. Consequently, if $\{h(x,\nabla u_k)\}$ is relatively weakly compact in $L^1(\Omega)$, then the limit of a (sub)sequence can be fully described by the Young measure generated by $\{\nabla u_k\}$, see *e.g.* [2,28,29]. Hence, $\hat{\nu}$ is supported on $\mathbb{R}^{m\times n}$. □

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