On a Pole Assignment by State Feedback in Non-square Linear Systems

Tetiana KOROTKA

8. 11. 2010

Contents

Introduction

Consider a linear, time-invariant system (E, A, B):

$$E\dot{x}(t) \;=\; Ax(t) + Bu(t),\ t \geq 0$$

where
- $E, A \in \mathbb{R}^{q\times n},\ B \in \mathbb{R}^{q\times m},\ \operatorname{rank} B = m$

Applying **the state feedback**

$$u(t) \;=\; Fx(t) + v(t),$$

where
- $F \in \mathbb{R}^{m\times n}$, and $v(t)$ is a new external input

gives the closed-loop system $(E, A + BF, B)$:

$$E\dot{x}(t) \;=\; (A + BF)x(t) + Bv(t),\quad t \geq 0$$

Motivation

change $\boldsymbol{F}$ in $(E, A + BF, B)$

$\Downarrow$

modify the dynamical behavior of (E, A, B):

the pole structure of the system

Problems:

- **pole structure assignment** (PSA)
- **pole assignment** (PA)

$\downarrow$

Non-square system (E, A, B)

Basic definitions: pole structure, regularizable system

Definition

The pole structure of the system (E, A, B) is defined by the **zero structure of the pencil $sE - A$**.

the finite zero structure

the invariant polynomials of $sE - A$

the infinite zero structure

the negative powers of s in the Smith-McMillan form at ∞ of $sE - A$

- (E, A, B) is called *regularizable* if $\exists$ a state feedback: $sE - A - BF$ is regular $\Leftrightarrow$ rank$(sE - A - BF)$ is full.

Problem Formulation

Problem Formulation (PSA)

Given

- a system (E, A, B)
- monic polynomials $\psi_1(s) \triangleright \psi_2(s) \triangleright \ldots \triangleright \psi_r(s)$
- integers $d_1 \geq d_2 \geq \ldots \geq d_{k_d}$.

Under what conditions there exists a state feedback :

the polynomials $\psi_i(s)$ and integers d_i will be

the invariant polynomials and infinite zero orders of $sE - A - BF$.

Pole assignment (PA) = **characteristic polynomial assignment** (regularizable system)

The previous results

The PSA and PA problems have been widely studied in the **square systems** ($q = n$).

1. Rosenbrock, 1970 (the seminal work) - explicit* and controllable system.
   (E is invertible)
2. Zaballa, 1988 - explicit and uncontrollable system.
3. Zagalak, Loiseau, 1992 - implicit* and controllable system.
   (E is singular)
4. Loiseau, Zagalak, 2009 - regularizable system.
   (PA + necessary conditions for the PSA).

Feedback Canonical Form

The feedback group (P, Q, G, F)

- $P,\ Q,\ G$ are invertible matrices over $\mathbb{R}$
- $F\ \in \mathbb{R}^{m\times n}$

Feedback canonical form (FCF) :

$$(P, Q, G, F) \circ (E, A, B) = (PEQ, P(A + BF)Q, PBG) \quad =:$$

$$=: (E_C,\ A_C,\ B_C)$$

$(sE_C - A_C) := \text{blockdiag}\{sE_{\alpha_i} - A_{\alpha_i}\}$, $\alpha = \epsilon, \sigma, q, p, l, \eta,\ i = 1, 2, \ldots, k_\alpha$

(1) $$\left.\overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & s & -1 \end{bmatrix}}^{\epsilon_i+1}\right\}\epsilon_i$$

(2) $$\left.\overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & s \end{bmatrix}}^{\sigma_i}\right\}\sigma_i$$

(3) $$\left.\overbrace{\begin{bmatrix} -1 & & \\ s & \ddots & \\ & \ddots & -1 \\ & & s \end{bmatrix}}^{q_i}\right\}q_i + 1$$

(4) $$\left.\overbrace{\begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix}}^{p_i+1}\right\}p_i + 1$$

(5) $$\left.\overbrace{\begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -a_{i0} & -a_{i1} & \cdots & s-a_{li} \end{bmatrix}}^{l_i}\right\}l_i$$

(6) $$\left.\overbrace{\begin{bmatrix} s & & \\ -1 & \ddots & \\ & \ddots & s \\ & & -1 \end{bmatrix}}^{\eta_i}\right\}\eta_i + 1$$

The form of B_C, indices in FCF

Matrix $\boldsymbol{B_C} := \mathrm{blockdiag}\,\{0, B_\sigma, B_q, 0, 0, 0\}$, where

$$B_\sigma := \mathrm{blockdiag}\left\{[0 \ldots 0\ 1]^T \in \mathbb{R}^{\sigma_i}\right\}$$

$$B_q := \mathrm{blockdiag}\left\{[0 \ldots 0\ 1]^T \in \mathbb{R}^{q_i+1}\right\}$$

The quantities describing the blocks:

1. the **nonproper** indices, $\epsilon_1 \geq \ldots \geq \epsilon_{k_\epsilon} \geq 0$;
2. the **proper** indices, $\sigma_1 \geq \ldots \geq \sigma_{k_\sigma} > 0$;
3. the **almost proper** indices, $q_1 \geq \ldots \geq q_{k_q} \geq 0$;
4. the **almost nonproper** indices, $p_1 \geq \ldots \geq p_{k_p} > 0$;
5. the **fixed invariant** polynomials $\alpha_1(s) \triangleright \alpha_2(s) \triangleright \cdots \triangleright \alpha_{k_l}(s)$, $\alpha_i(s) = s^{l_i} + a_{il_i} s^{l_i - 1} + \cdots + a_{i1} s + a_{i0}$;
6. the **row minimal** indices of $[sE_C - A_C,\ -B_C]$, $\eta_1 \geq \ldots \geq \eta_{k_\eta} \geq 0$.

Normal External Description(NED)

Definition

The matrices $N(s), D(s)$ are said to form a **NED** of the system (E, A, B) if they satisfy the following conditions:

- $[sE - A \quad -B] \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0$
  where $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ forms a minimal polynomial basis for $\mathrm{Ker}[sE - A \quad -B]$
- $\Pi[sE - A]N(s) = 0$
  where Π is a maximal annihilator of B,

  $N(s)$ forms a minimal polynomial basis for $\mathrm{Ker}\Pi[sE - A]$.

The extension of the system

The NED of (E_C, A_C, B_C) reflects information:

+ $\epsilon_i, \ \sigma_i$

* controllability indices of regularizable (E, A, B)

$=: c_i, \ i = 1, 2, \ldots, k_\epsilon + k_\sigma$

System is controllable iff $\sum c_i = \operatorname{rank} E$

− $q_i, \ p_i, \ \eta_i, \ \alpha_i(s)$ (the hidden part of the system)

How to include the hidden part?

B_C is extended: $[B_C, \overline{B}_C]$

$\Downarrow$

the hidden part of (E_C, A_C, B_C) appears in the **NED**

$(E_C, A_C, [B_C \ \overline{B}_C])$ – the extended system of (E_C, A_C, B_C)

Conformal mapping

To deal with finite and infinite zeros in a unified way:

conformal mapping $s = \frac{(1+aw)}{w}$

where
- $a \in \mathbb{R}$, $a \neq 0$, and is not a pole of the system.

the **infinite** zero structure of $sE_C - A_C$ = the **finite** zero structure of $w\tilde{E}_C - \tilde{A}_C$ at $w = 0$

where $w\tilde{E} - \tilde{A}_C$ is the w-analogue of $sE_C - A_C$.

The action of the state feedback upon (E_C, A_C, B_C):

$$\begin{bmatrix} sE_C - A_C & -[B_C \ \bar{B}_C] \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ F & I_m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ -F & I_m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} N_E(s) \\ D_E(s) \end{bmatrix} = 0,$$

$$\begin{bmatrix} sE_C - A_C - \boldsymbol{B_C F} & -[B_C \bar{B}_C] \end{bmatrix} \begin{bmatrix} N_E(s) \\ D_{EF}(s) \end{bmatrix} = 0$$

$$\boldsymbol{D_{EF}(s) := \begin{bmatrix} D_C(s) - F N_C(s) & -F \overline{N}_C(s) \\ 0 & \overline{D}_C(s) \end{bmatrix}}$$

where • $\overline{N}_C(s)$, $\overline{D}_C(s)$ form the NED of the hidden part

The main property:

The non unit invariant factors of both

$w\tilde{E}_C - \tilde{A}_C - \tilde{B}_C(w)F$ and $\tilde{D}_{EF}(w)$ coincide for any F.

where • $\tilde{D}_{EF}(w)$ is a $w-$analogue of the $D_{EF}(s)$.

Description of the modification of a system by a state feedback

$$\tilde{D}_{EF}(w) := \left[\begin{array}{cccccc} \tilde{D}_{11} & S_{\tilde{\sigma}} + \tilde{D}_{12} & \tilde{D}_{13} & \tilde{D}_{14} & \tilde{D}_{15} & \tilde{D}_{16} \\ \tilde{D}_{21} & \tilde{D}_{22} & S_{\tilde{q}} + \tilde{D}_{23} & \tilde{D}_{24} & \tilde{D}_{25} & \tilde{D}_{26} \\ \hline 0 & 0 & \mathrm{diag}\{w^{q_i}\} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{diag}\{w^{p_i}\} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{\tilde{\alpha}} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{\tilde{\eta}} \end{array}\right]$$

$$S_{\tilde{\sigma}} := \mathrm{diag}\left\{(1+aw)^{\sigma_i}\right\}_{i=1}^{k_\sigma}, \qquad S_{\tilde{q}} := \mathrm{diag}\left\{(1+aw)^{q_i}\right\}_{i=1}^{k_q}$$

$$S_{\tilde{\alpha}} := \mathrm{diag}\left\{\tilde{\alpha}_i(w)\right\}_{i=1}^{k_l}, \qquad S_{\tilde{\eta}} := \mathrm{blockdiag}\left\{\begin{bmatrix}(1+aw)^{\eta_i} \\ -w^{\eta_i}\end{bmatrix}\right\}_{i=1}^{k_\eta}$$

and $D_{ij}(s)$ are arbitrary matrices satisfying conditions

$$\deg_{ci}\begin{bmatrix} D_{1j}(s) \\ D_{2j}(s)\end{bmatrix} \leq j_i, \;\; j = \epsilon, \sigma, q, p, l, \eta.$$

Under what conditions there exists a state feedback : the full (row or column) rank pencil $sE - A - BF$?

Conditions of solvability of PA:

- (a) full row rank iff $k_\epsilon \geq k_q$ and $k_\eta = 0$
- (b) full column rank iff $k_q \geq k_\epsilon$

If (a) & (b) $\Rightarrow$ system is regularizable

If (a) $\oplus$ (b) $\Rightarrow$ *system is weakly (row or column) regularizable*

$\Downarrow$

(at least) one of the principal minors (of order $\min\{q, n\}$) $\neq \mathbf{0}$

$\Downarrow$

Pole assignment (PA) $=$ **the assignment of the greatest common divisor of the principal minors (gcdpm)** of $sE - A - BF$

The illusration of the Proposition

Example

Let

$$[sE - A \quad -B] := \begin{bmatrix} s & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & s & -1 \end{bmatrix}$$

Defining $F = [1 \ 0 \ 0 \ 0]$, the pencil

$$sE - A - BF = \begin{bmatrix} s & -1 & 0 & 0 & 0 \\ 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & s \end{bmatrix}$$

is of full row rank.

Pole assignment in weakly regularizable system

Problem formulation(PA)

Given

- a weakly regularizable system (E, A, B)
- a monic polynomial $\psi(s)$
- an integer d

Under what conditions there exists a state feedback :

$$\tilde{\psi}(w)w^d \text{ will be a gcdpm } \left(w\tilde{E} - \tilde{A} - F\tilde{B}(w)\right)?$$

• Regularizable system $(k_\epsilon = k_q \text{ and } \; k_\eta = 0)$

$$\deg \psi(s) + d = \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i$$

$$\psi(s) \triangleright \alpha_1(s)\alpha_2(s)...\alpha_{k_l}(s)$$

$$d \geq \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i \ .$$

if $k_\epsilon = 0$:

$$\deg \psi(s) = \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_l} l_i$$

- the quantities $\alpha_i(s), p_i, q_i$ can not be changed by state feedback
- the sum of the indices ϵ_i, σ_i is the number of the poles that can be freely assigned either to finite or infinite locations

• Row regularizable system $(k_\epsilon \geq k_q \text{ and } \; k_\eta = 0)$

$$\tilde{D}_{EF}(w) \cong \left[\begin{array}{cccccc} \star & \tilde{D}_{11}' & \tilde{D}_{12} & \tilde{D}_{13} & \tilde{D}_{14} & \tilde{D}_{15} \\ \star & \tilde{D}_{21}' & \tilde{D}_{22} & \tilde{D}_{23} & \tilde{D}_{24} & \tilde{D}_{25} \\ \hline 0 & 0 & 0 & \mathrm{diag}\{w^{q_i}\} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{diag}\{w^{p_i}\} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{\tilde{\alpha}} \end{array}\right]$$

$$\tilde{\boldsymbol{\psi}}(\boldsymbol{w}) = \tilde{\boldsymbol{\psi}}'(\boldsymbol{w})\boldsymbol{w}^{(\boldsymbol{q_i}+\boldsymbol{p_j})}\boldsymbol{S_{\tilde{\alpha}}}, \quad i = 1, \dots, k_q, \; j = 1, \dots, k_p$$

where $\tilde{\psi}'(w) := \mathrm{gcdpm}\begin{bmatrix} \star & \tilde{D}_{11}' & \tilde{D}_{12} \\ \star & \tilde{D}_{21}' & \tilde{D}_{22} \end{bmatrix}$ and det $\begin{bmatrix} \tilde{D}_{11}' & \tilde{D}_{12} \\ \tilde{D}_{21}' & \tilde{D}_{22} \end{bmatrix} \neq 0$

$$0 \quad \leq \quad \deg \psi'(w) \quad \leq \quad \sum_{i=1}^{k_q} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i$$

$$\Downarrow$$

represents the sum of the controllable poles

• Row regularizable system $(k_\epsilon \geq k_q \text{ and } k_\eta = 0)$

Necessary conditions:

$$\deg \psi(s) + d \leq \sum_{i=1}^{k_q} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i$$

$$\psi(s) \triangleright \alpha_1(s)\alpha_2(s)\cdots\alpha_{k_l}(s)$$

$$d \geq \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i$$

Example

Let $\epsilon_1 = 0$ and $\sigma_1 = 3$. The matrix $D_{EF}(s)$ is of the form

$$D_{EF}(s) = \left[\begin{array}{cc} \alpha_0 & s^3 + \beta_2 s^2 + \beta_1 s + \beta_0 \end{array} \right]$$

$\Rightarrow$ the degrees of a principal minor are either $\mathbf{0}$ or $\mathbf{3}$,
but never $\mathbf{1}$ or $\mathbf{2}$
(although they satisfy the condition $\deg \psi(s) \leq 3$).

• Column regularizable system $(k_q \geq k_\epsilon)$

$$\tilde{D}_{EF}(w) \cong \left[\begin{array}{cccccc} \tilde{D}'_{11} & \tilde{D}'_{12} & \tilde{D}'_{13} & \tilde{D}'_{14} & \tilde{D}'_{15} & 0 \\ \tilde{D}'_{21} & \tilde{D}'_{22} & \tilde{D}'_{23} & \tilde{D}'_{24} & \tilde{D}'_{25} & 0 \\ \hline 0 & 0 & \operatorname{diag}\{w^{q'_i}\} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{diag}\{w^{p_i}\} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{\tilde{\alpha}} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{k_\eta + k_q - k_\epsilon} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where $\{q'_i\}$ be a subset of the indices q_i, $k_{q'} := \operatorname{card}\{q'_i\} = k_\epsilon$

Necessary and sufficient conditions:

$$\begin{aligned} \deg \psi(s) + d &= \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_{q'}} q'_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i \\ \psi(s) &\triangleright \alpha_1(s)\alpha_2(s)\cdots\alpha_{k_l}(s) \\ d &\geq \sum_{i=1}^{k_{q'}} q'_i + \sum_{i=1}^{k_p} p_i \end{aligned}$$

Maximal number of zeros (including multiplicities)

Theorem

Given *a weakly regularizable system* (E, A, B), *a monic polynomial* $\psi(s)$, *an integer* d.

$\Rightarrow$ *there exists a state feedback :* $\psi(s)$ *and* d *will be the gcdpm and the sum of the infinite zero orders of* $sE-A-BF$ *iff:*

$$\deg\psi(s) + d = \sum_{i=1}^{k_r}\epsilon_i + \sum_{i=1}^{k_\sigma}\sigma_i + \sum_{i=1}^{k_r} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i$$

$$\psi(s) \triangleright \alpha_1(s)\alpha_2(s)\dots\alpha_{k_l}(s)$$

$$d \geq \sum_{i=1}^{k_r} q_i + \sum_{i=1}^{k_p} p_i$$

where $r := \min\{k_\epsilon, k_q\}$.

Some Remarks:

- The number of zeros that can be assigned by a state feedback is not increased by the redundant ϵ, q- and η-blocks
- In the row regularizable systems the presence of the redundant ϵ-blocks may lead to the cancellation of all poles, which are assignable at our wil
- These ϵ-blocks represent the so called 'internal degree of freedom' of the system which can not be influenced by a control input. Using a state feedback, the influence of this degree of freedom could be spread on the controllable part of the system.
- The quantities η_i and the redundant indices q_i present the constraints on the solution $x(t)$ of the system.