Symmetry

Tomáš Suk

Symmetry

Symmetry in art - harmonious or aesthetically pleasing proportionality and balance

Symmetry in physics - quantum symmetry, more general

Symmetry in chemistry, geology, biology - only approximate

Symmetry in mathematics - today's theme

Symmetry in mathematics

Function $f(\mathbf{x})$ is symmetric, if there is such geometric transformation G that

$$f(\mathbf{x}) = f(G(\mathbf{x}))$$

$f(\mathbf{x})$ is then symmetric with respect to G

$\mathbf{x}$ – coordinates in n-dimensional space

Symmetry in mathematics

$$f(\mathbf{x}) = f(G(\mathbf{x}))$$

Two limit cases:

$f(\mathbf{x})$ constant, $G(\mathbf{x})$ arbitrary – uninteresting

$f(\mathbf{x})$ arbitrary, $G(\mathbf{x})$ identity – no symmetry

More than one $G(\mathbf{x})$:

They create a group -

operation: composition

neutral element: identity

Class of symmetry

- Set of similar groups, they differ by a parameter only
- Each group is applicable on different function

Terminological remark:

symmetry group ≠ symmetric group

Symmetry in 1D

- Reflection symmetry σ, C_2, D_1 $f(a-x)=f(x-a)$
  - also mirror symmetry

- Translational symmetry Z $f(x)=f(x+k\lambda)$

- Reflection and translation D_∞

Symmetry in 1D

- Scale symmetry $f(x)=f(b^k x)$

$\rightarrow$ fractals

Symmetry in 2D

- Reflection symmetry = axial symmetry
- Rotational symmetry – fold number *n*

  $C_1, C_2, C_3, C_4, \dots$ rotation by 360°/n
- Dihedral symmetry – reflection + rotation

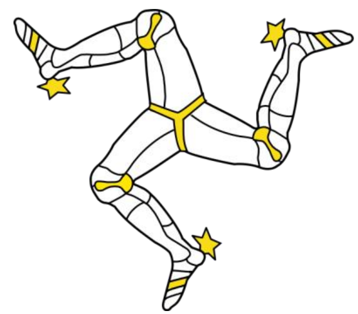
  $D_1, D_2, D_3, D_4, \dots$
- Circular symmetry $D_\infty = O(2)$

  – reflection + rotation by arbitrary angle

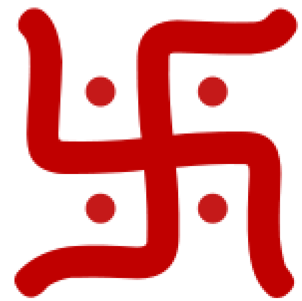
Rotational symmetry in 2D

Examples

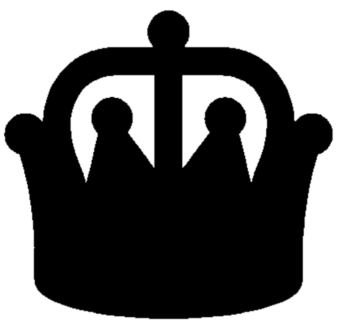
C_3



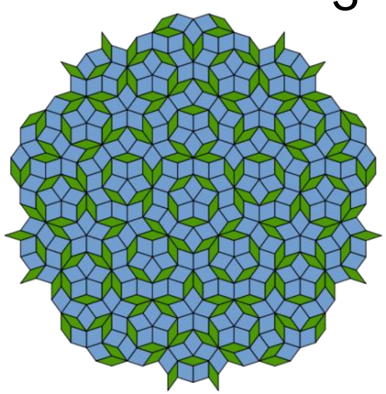
C_3



C_4



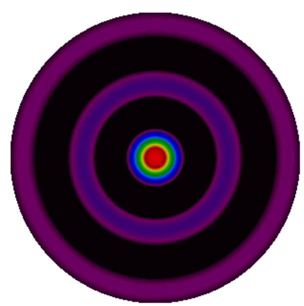
D_1



D_5

D_6

D_{24}



D_∞

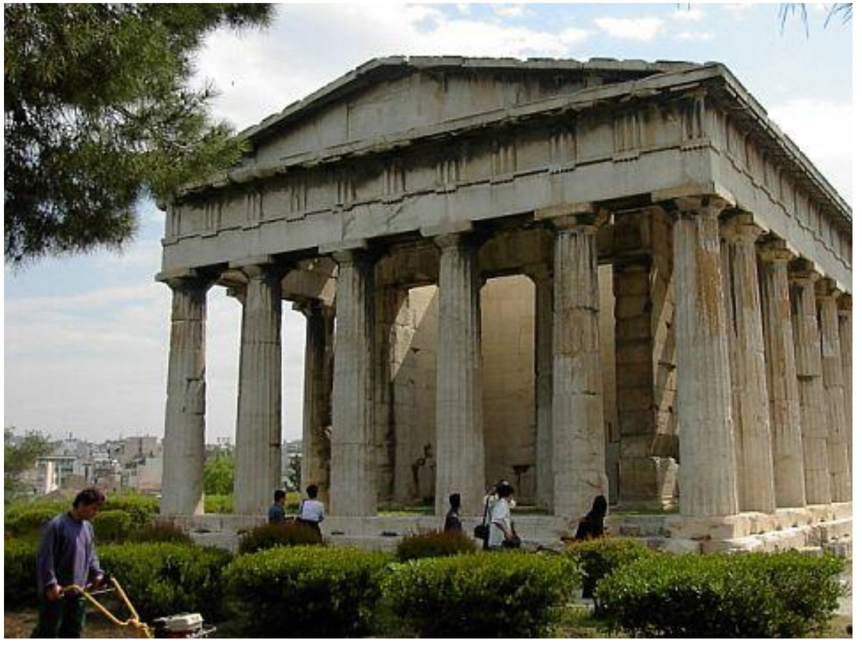
Rotation + reflection in 2D

- C_1, C_2, C_3, C_4, …
- D_1, D_2, D_3, D_4, … D_∞

C1 - No symmetry

Frieze symmetry

- $f(x,y)$, x – infinite support, y – finite support
- Frieze - long stretch of painted, sculpted or even calligraphic decoration



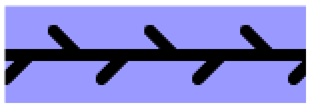
English: frieze →← freeze

Czech: vlys →← vlis

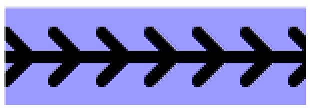
Frieze symmetry

1. 

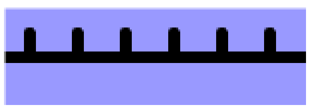
p111 (translation only)

2. 

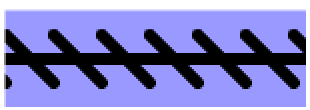
p1a1 (translation + glide reflection)

3. 

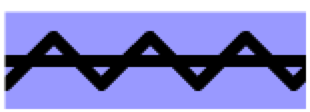
p1m1 (translation + horizontal line reflection + glide reflection)

4. 

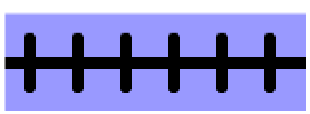
pm11 (translation + vertical line reflection)

5. 

p112 (translation + 180° rotation)

6. 

pma2 (translation + 180° rotation + vertical line reflection + glide reflection)

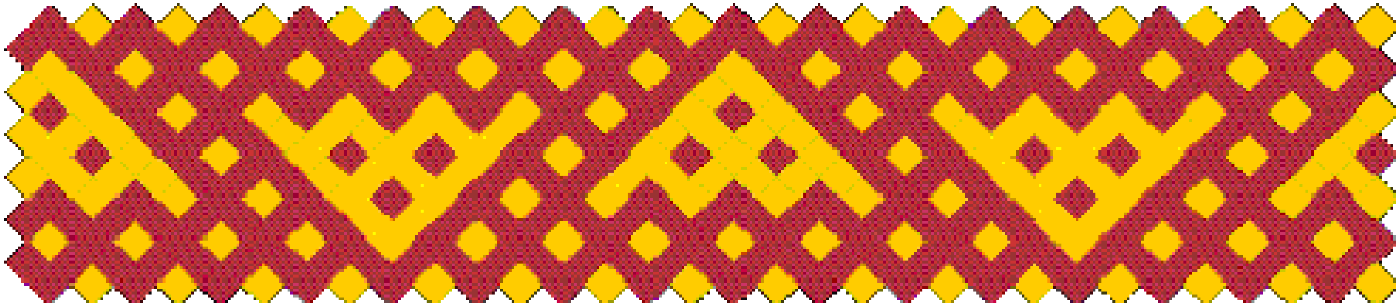
7. 

pmm2 (translation + 180° rotation + horizontal line reflection + vertical line reflection + glide reflection)

Glide reflection

- Translation & reflection

$$f(x,y)=f(x+\lambda,y)$$
$$f(x,y)=f(x+\lambda/2,-y)$$



Wallpaper symmetry

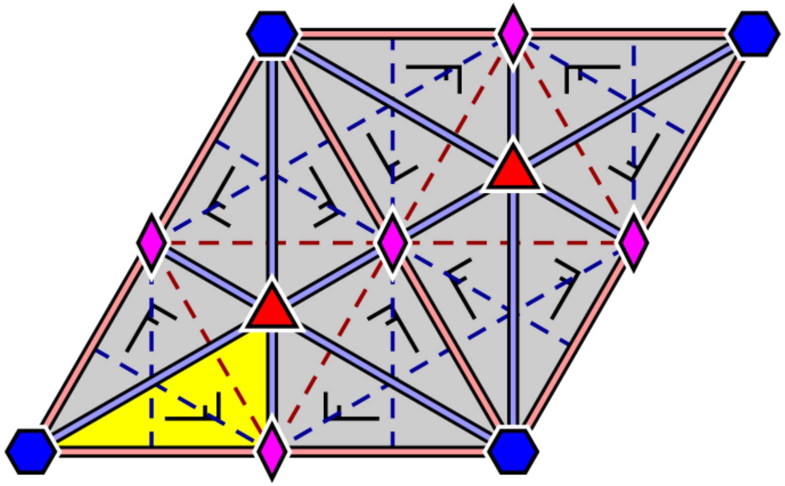
- Also ornamental symmetry
- Translation in >1 directions + reflection + rotation
- Fold number 1,2,3,4,6
- 17 groups: p1, p2, pm, pg, cm, pmm, pmg, pgg, cmm, p4, p4m, p4g, p3, p3m1, p31m, p6, p6m

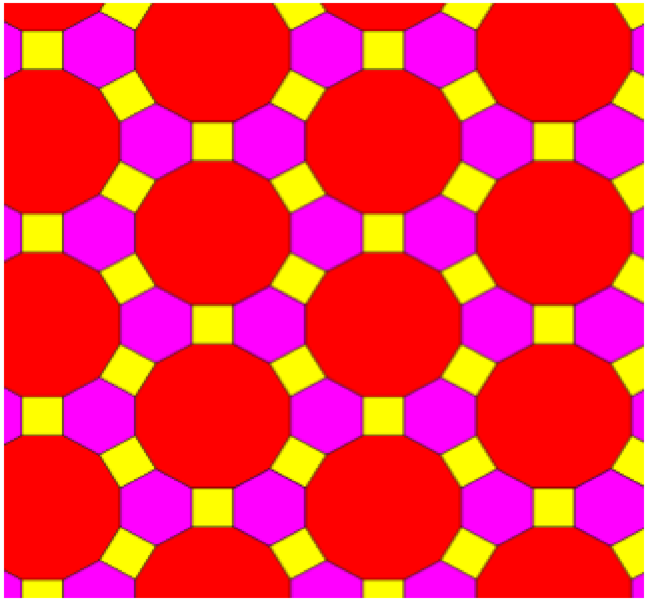
Wallpaper symmetry

<table>
<tr><th rowspan="2">Size of smallest rotation</th><th colspan="6">Has reflection?</th></tr>
<tr><td colspan="4">Yes</td><td colspan="2">No</td></tr>
<tr><td>360° / 6</td><td colspan="4">p6m</td><td colspan="2">p6</td></tr>
<tr><td rowspan="2">360° / 4</td><th colspan="4">Has mirrors at 45°?</th><td colspan="2" rowspan="2">p4</td></tr>
<tr><td colspan="2">Yes: p4m</td><td colspan="2">No: p4g</td></tr>
<tr><td rowspan="2">360° / 3</td><th colspan="4">Has rot. centre off mirrors?</th><td colspan="2" rowspan="2">p3</td></tr>
<tr><td colspan="2">Yes: p31m</td><td colspan="2">No: p3m1</td></tr>
<tr><td rowspan="4">360° / 2</td><th colspan="4">Has perpendicular reflections?</th><th colspan="2" rowspan="2">Has glide reflection?</th></tr>
<tr><td colspan="3">Yes</td><td>No</td></tr>
<tr><th colspan="3">Has rot. centre off mirrors?</th><td rowspan="2">pmg</td><td rowspan="2">Yes: pgg</td><td rowspan="2">No: p2</td></tr>
<tr><td>Yes: cmm</td><td colspan="2">No: pmm</td></tr>
<tr><td rowspan="2">none</td><th colspan="4">Has glide axis off mirrors?</th><th colspan="2">Has glide reflection?</th></tr>
<tr><td colspan="2">Yes: cm</td><td colspan="2">No: pm</td><td>Yes: pg</td><td>No: p1</td></tr>
</table>

Wallpaper symmetry

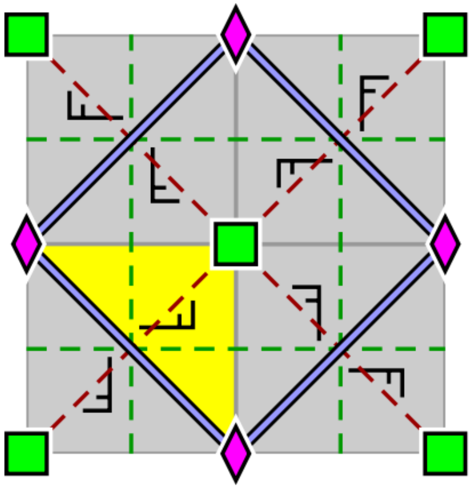
Example: p6m



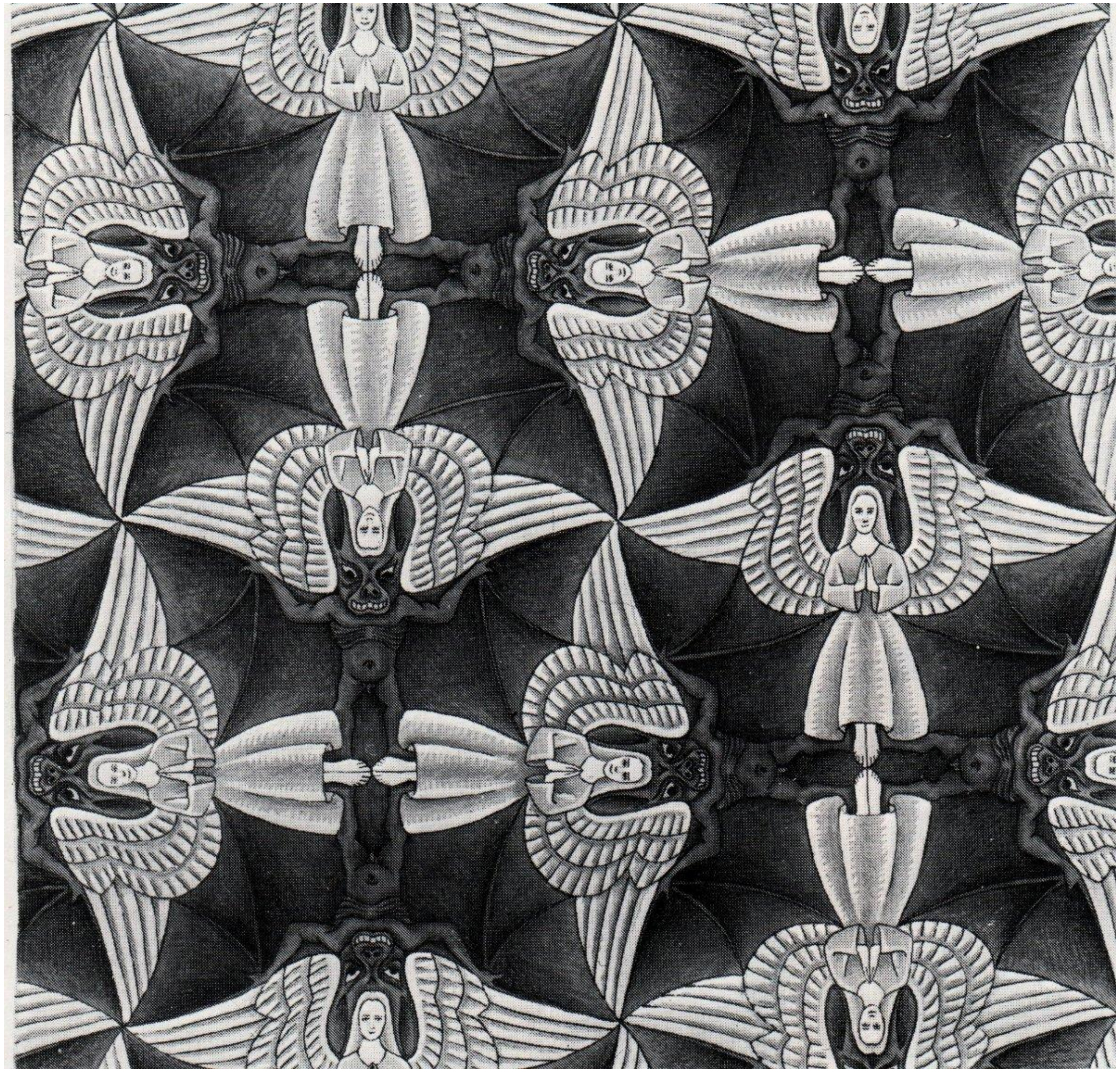


Wallpaper symmetry

Example: p4g

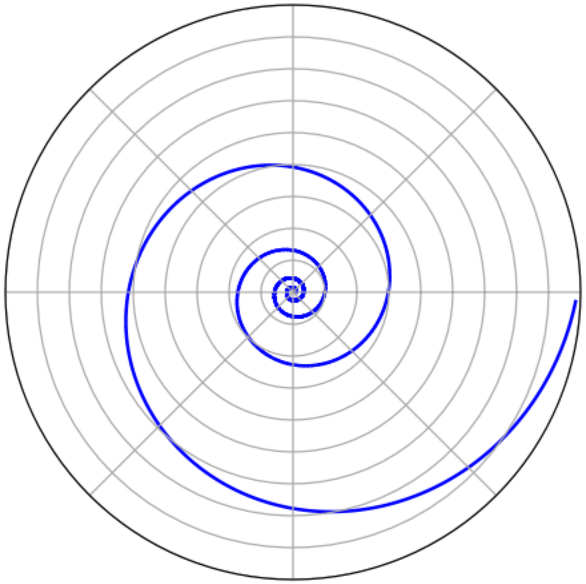


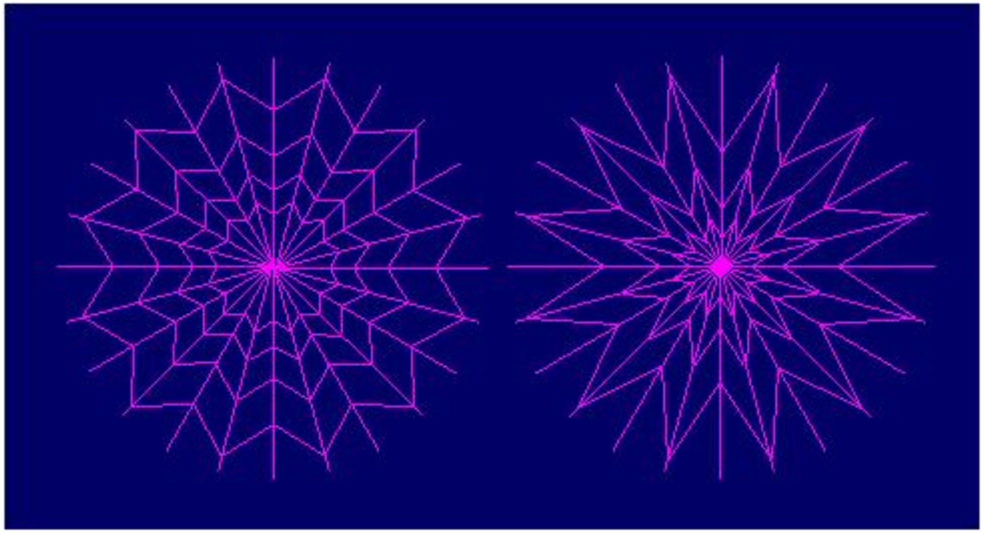
Maurits Cornelis Escher:
Angels and devils

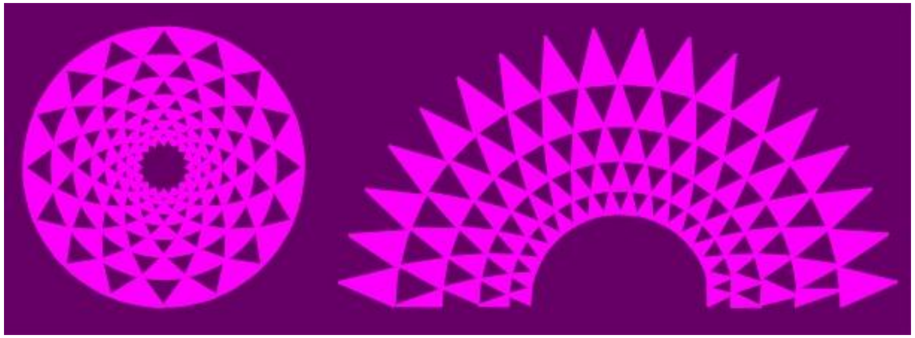


Similarity Symmetry

- Scaling & rotation
- Scaling & translation





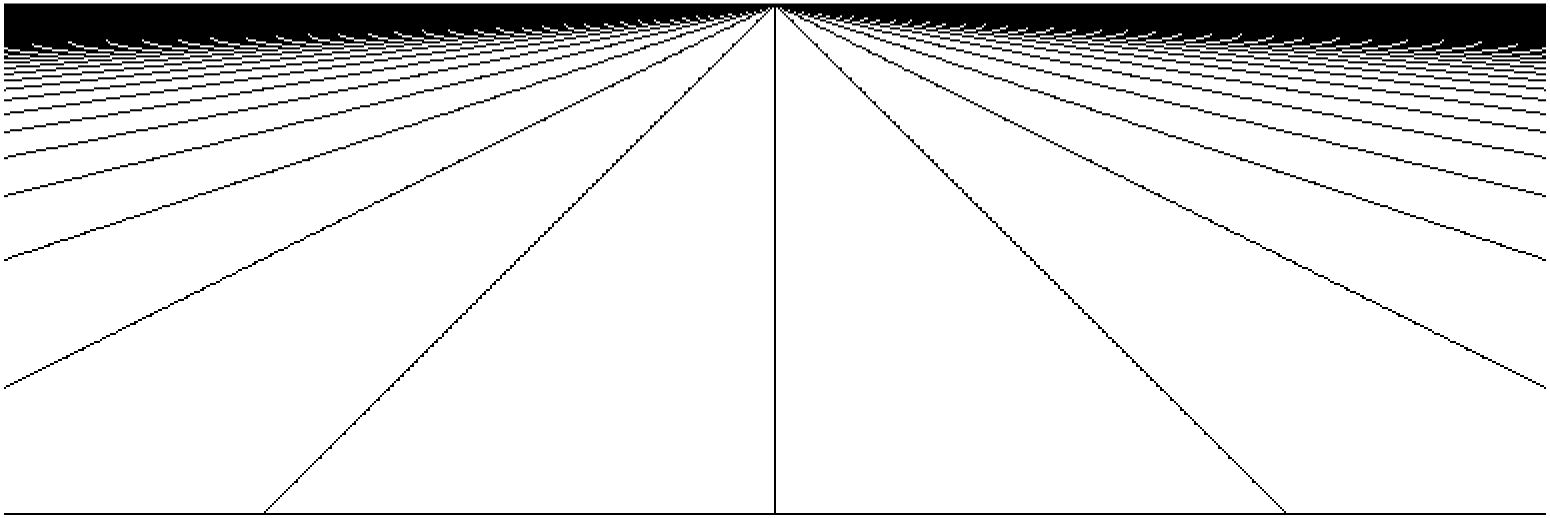


Similarity Symmetry

Nautilus pompilius

Other symmetries ?

Symmetry to skew $f(x,y)=f(x+k\lambda y,y)$



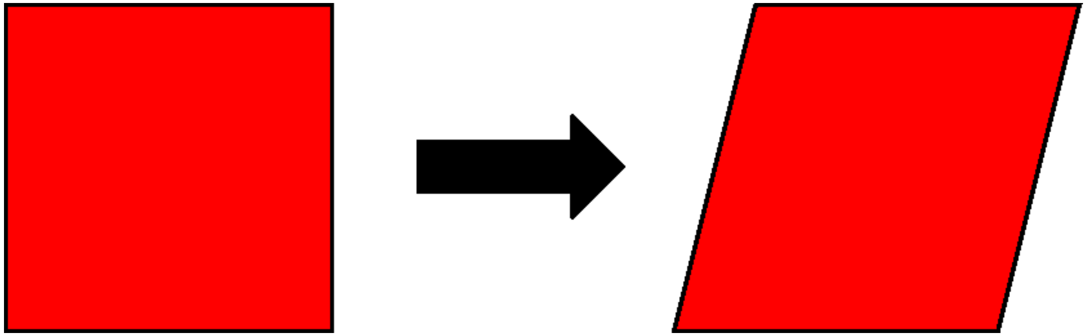
Terminological remark:

symmetry to skew ≠ skew symmetry

$f(x,y)= -f(y,x)$

Unusual symmetries

Symmetry to skew $f(x,y)=f(x+k\lambda y,y)$

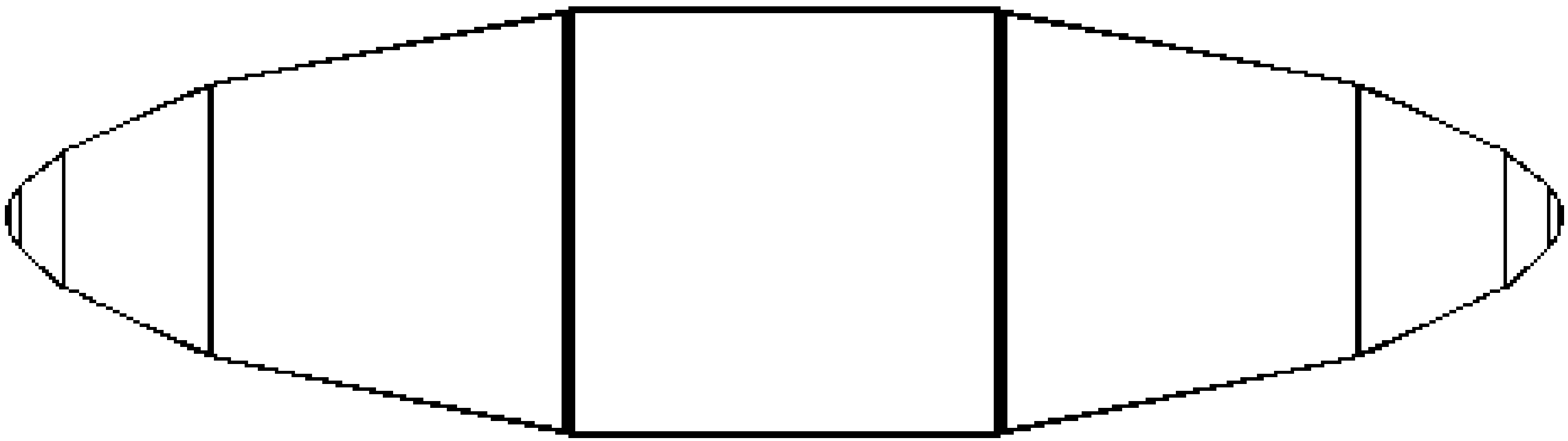


Unusual symmetries

Projective symmetry

$$x' = \frac{a_0 + a_1 x + a_2 y}{c_0 + c_1 x + c_2 y}$$

$$y' = \frac{b_0 + b_1 x + b_2 y}{c_0 + c_1 x + c_2 y}$$



Unusual symmetries

Projective symmetry

Unusual symmetries

Symmetry to Möbius transform

– projection of sphere to plane

Complex plane

Real plane

$$z = x + yi$$

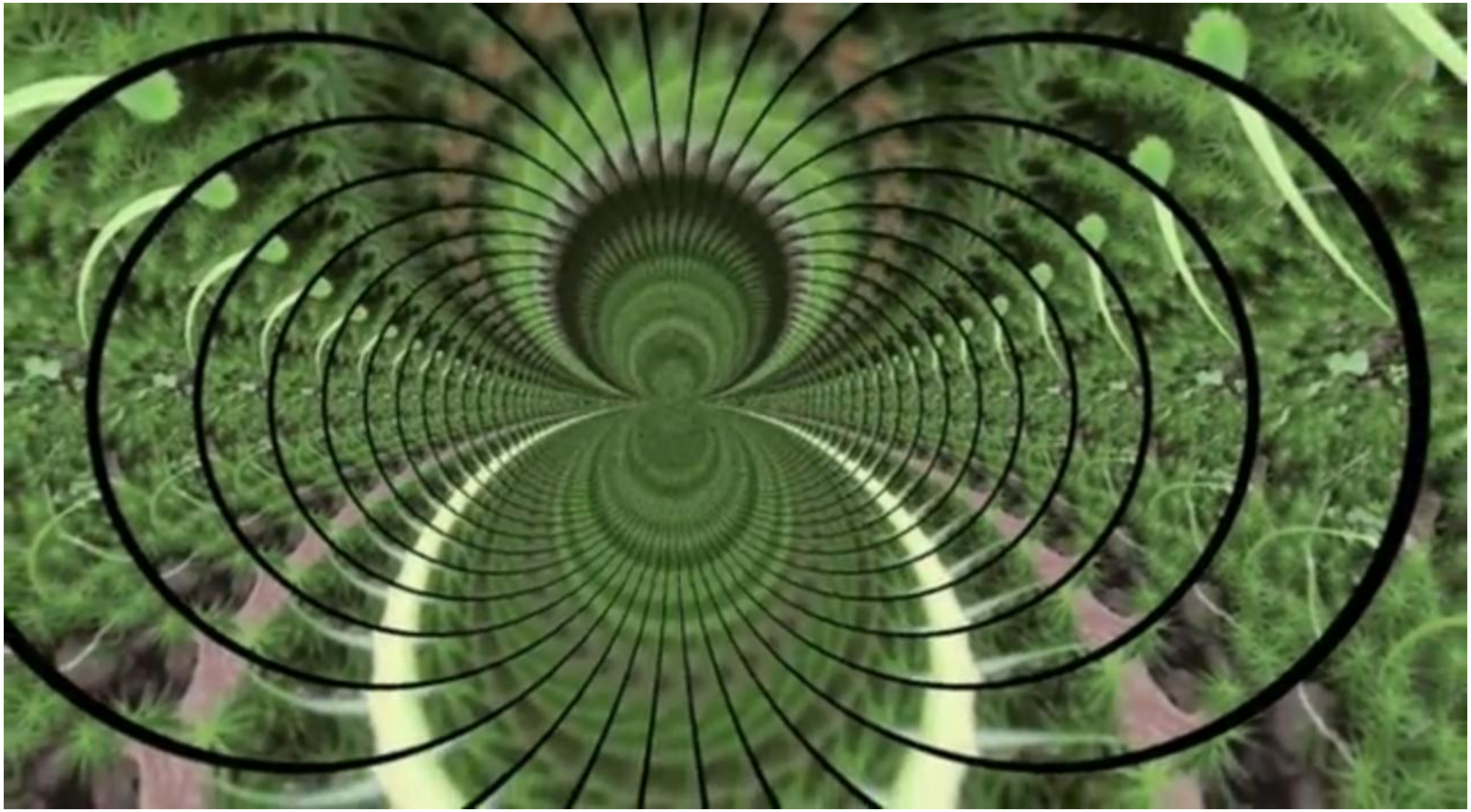
$$z' = \frac{az + b}{cz + d}$$

$$x' = \frac{ac(x^2 + y^2) + (ad + bc)x + bd}{c^2(x^2 + y^2) + 2cdx + d^2}$$

$$y' = \frac{(ad - bc)y}{c^2(x^2 + y^2) + 2cdx + d^2}$$

Unusual symmetries

Symmetry to Möbius transform



Symmetry in 3D

- Reflection symmetry – plane of symmetry
- Rotational symmetry – axis of symmetry, combination of more axes?
- Translational symmetry
- Others

Rotational symmetry in 3D

- 1 axis of *n*-fold rotational symmetry – pyramids

$$C_1, C_2, C_3, C_4, \ldots$$

- 1 axis of *n*-fold symmetry + *n* perpendicular axes of 2-fold symmetry – prism

$$D_1, D_2, D_3, D_4, \ldots$$

- Symmetrical polyhedra

T, O, I

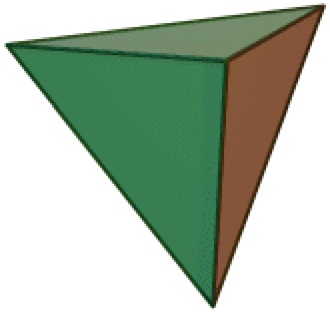
Rotational symmetry in 3D

Tetrahedron T

– 4 axes of 3-fold symmetry,

– 3 axes of 2-fold symmetry,

total fold number 12



Rotational symmetry in 3D

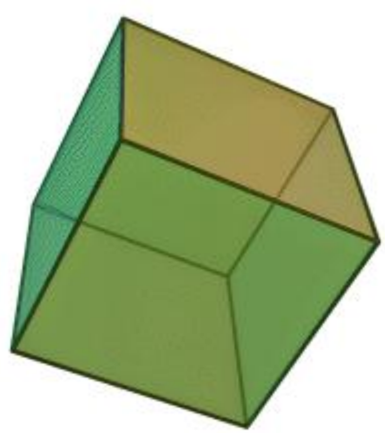
Cube + Octahedron O

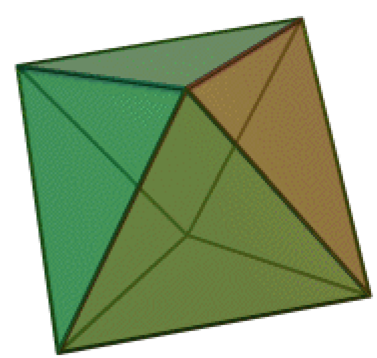
– 3 axes of 4-fold symmetry,

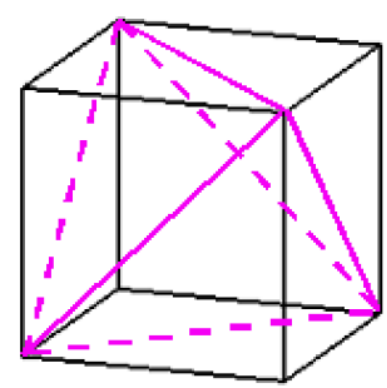
– 4 axes of 3-fold symmetry,

– 6 axes of 2-fold symmetry,

total fold number 24

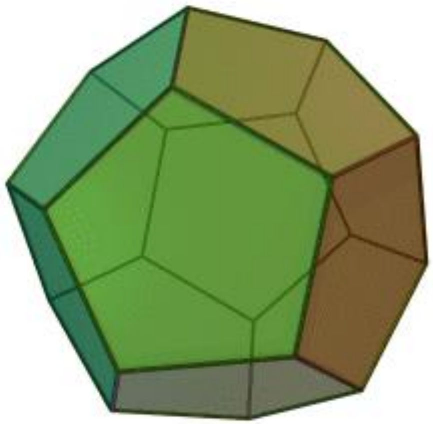


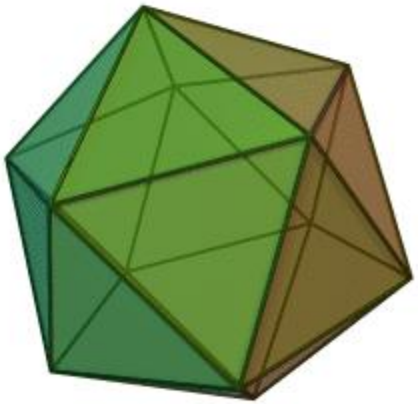




Rotational symmetry in 3D

Dodecahedron + icosahedron I
– 6 axes of 5-fold symmetry,
– 10 axes of 3-fold symmetry,
– 15 axes of 2-fold symmetry,
total fold number 60





Infinite rotational symmetry in 3D

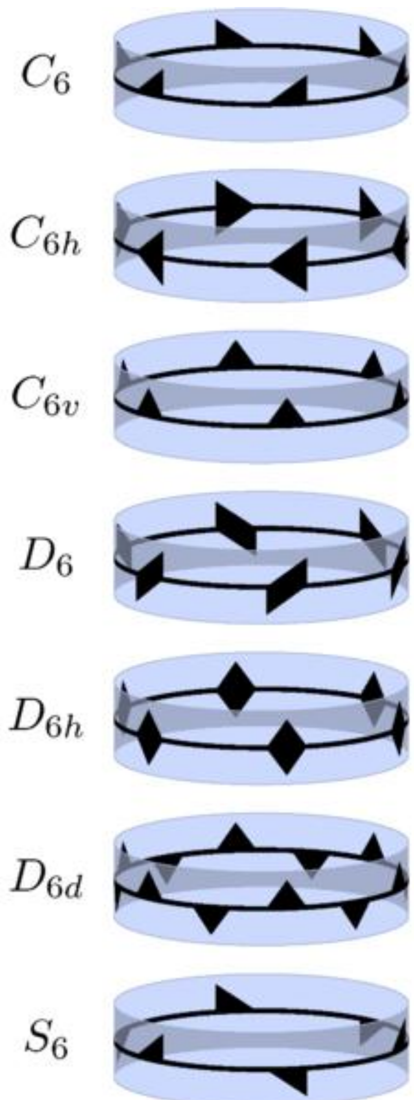
- 1 axis of ∞-fold rotational symmetry
  - conic C_∞ e.g. bottle
- 1 axis of ∞-fold symmetry + ∞ axes of 2-fold symmetry – cylinder D_∞
- ∞ axes of ∞-fold symmetry
  - sphere $K=O(3)$

Rotation + reflection in 3D

- C_n – n-fold rotational symmetry
- C_{nh} – C_n + horizontal reflection plane
- C_{nv} – C_n + n vertical reflection planes
- D_n – dihedral symmetry
- D_{nh} – D_n + horizontal reflection plane
- D_{nd} – D_n + S_{2n}
- S_{2n} – rotation & reflection

$C_{1h}=C_{1v}$ $D_1=C_2$ $D_{1h}=C_{2v}$ $D_{1d}=C_{2h}$

Rotation + reflection in 3D



C_{nv} – Pyramidal symmetry

D_{nh} – Prismatic symmetry

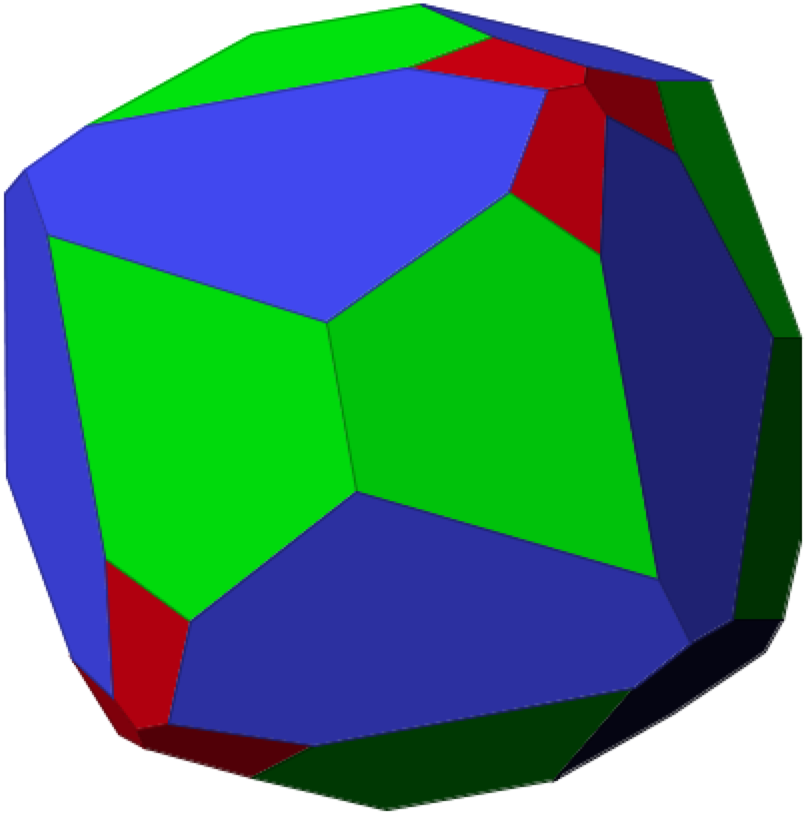
D_{nd} – Antiprismatic symmetry

Rotation + reflection in 3D

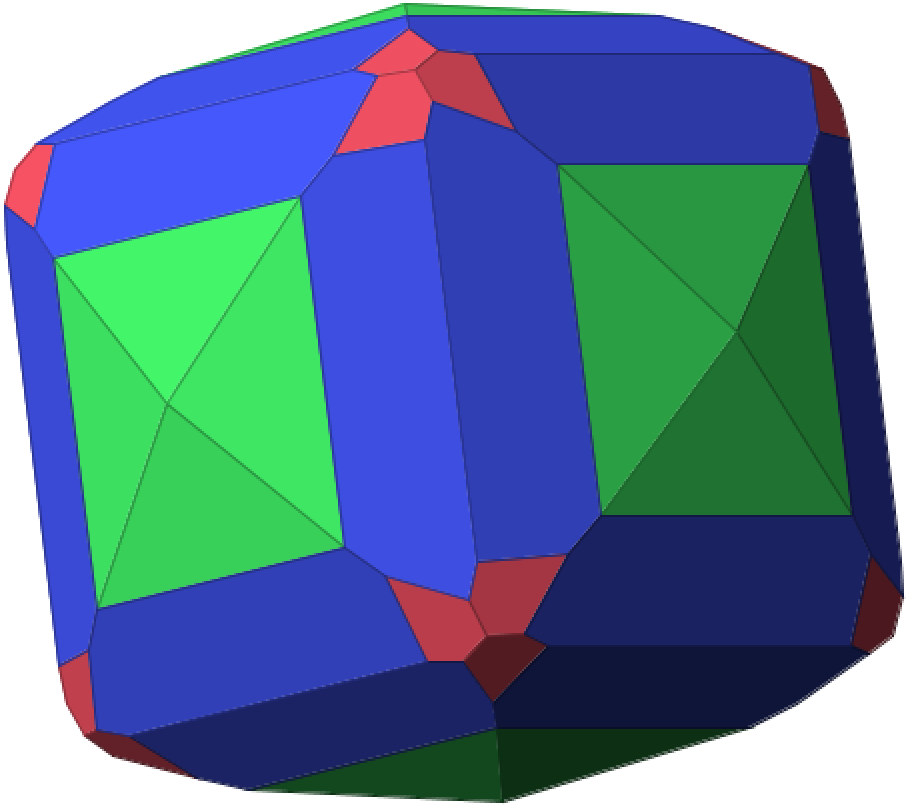
- T – chiral tetrahedral symmetry
- T_h – pyritohedral symmetry - 3 planes
- T_d – full tetrahedral symmetry - 6 planes
- O – chiral octahedral symmetry
- O_h – full octahedral symmetry - 9 planes
- I – chiral icosahedral symmetry
- I_h – full icosahedral symmetry - 15 planes

Rotation + reflection in 3D

Group T



Group O



No reflection symmetry

Rotation + reflection in 3D

- $C_1, C_2, C_3, C_4, \ldots$
- $C_{1h}, C_{2h}, C_{3h}, C_{4h}, \ldots$
- $C_{1v}, C_{2v}, C_{3v}, C_{4v}, \ldots$ $C_{\infty v}$
- $D_1, D_2, D_3, D_4, \ldots$
- $D_{1h}, D_{2h}, D_{3h}, D_{4h}, \ldots$ $D_{\infty h}$
- $D_{1d}, D_{2d}, D_{3d}, D_{4d}, \ldots$
- $S_2, S_4, S_6, S_8, \ldots$
- $T, T_h, T_d, O, O_h, I, I_h,$ K

Central symmetry

- 1D: $f(x)=f(-x)$

  reflection

- 2D: $f(x,y)=f(-x,-y)$

  2-fold rotational symmetry

- 3D: $f(x,y,z)=f(-x,-y,-z)$

  reflection & rotation by 180° = group S_2

Rotation + reflection + translation in 3D

- 7 crystal systems
- 32 point groups
- 14 Bravais lattices
- 230 space groups

Crystal systems

	Fold number	Point groups
• triclinic	n=1	C_1, S_2
• monoclinic	n=2	C_2, C_{1h}, C_{2h}
• orthorhombic	n=2	C_{2v}, D_2, D_{2h}
• trigonal	n=3	C_3, C_{3v}, D_3, D_{3d} , S_6
• tetragonal	n=4	C_4, C_{4h}, C_{4v} , D_4, D_{4h} , D_{2d}, S_4
• cubic	n=3,4	T, T_h, T_d, O, O_h
• hexagonal	n=6	C_6, C_{3h}, C_{6h} , C_{6v}, D_6, D_{3h}, D_{6h}

Space groups - Schönflies symbols

- triclinic C_1^1, S_2^1
- monoclinic $C_2^{1\text{-}3}$, $C_{1h}^{1\text{-}4}$, $C_{2h}^{1\text{-}6}$
- orthorhombic $C_{2v}^{1\text{-}22}$, $D_2^{1\text{-}9}$, $D_{2h}^{1\text{-}28}$
- trigonal $C_3^{1\text{-}4}$, $C_{3v}^{1\text{-}6}$, $D_3^{1\text{-}7}$, $D_{3d}^{1\text{-}6}$, $S_6^{1\text{-}2}$
- tetragonal $C_4^{1\text{-}6}$, $C_{4h}^{1\text{-}6}$, $C_{4v}^{1\text{-}12}$, $D_4^{1\text{-}10}$, $D_{4h}^{1\text{-}20}$, $D_{2d}^{1\text{-}12}$, $S_4^{1\text{-}2}$
- cubic $T^{1\text{-}5}$, $T_h^{1\text{-}7}$, $T_d^{1\text{-}6}$, $O^{1\text{-}8}$, $O_h^{1\text{-}10}$
- hexagonal $C_6^{1\text{-}6}$, C_{3h}^1, $C_{6h}^{1\text{-}2}$, $C_{6v}^{1\text{-}4}$, $D_6^{1\text{-}6}$, $D_{3h}^{1\text{-}4}$, $D_{6h}^{1\text{-}4}$

Bravais lattices

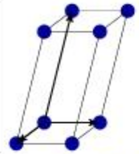
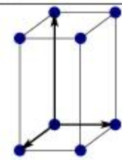
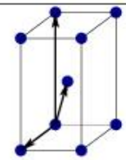
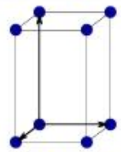
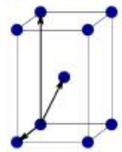
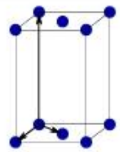
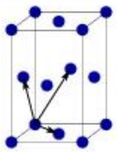
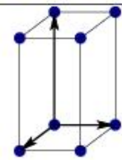
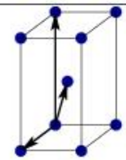
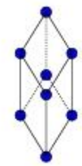
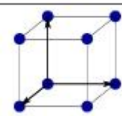
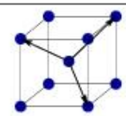
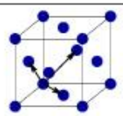
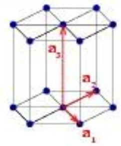
Bravais lattice	Parameters	Simple (P)	Volume centered (I)	Base centered (C)	Face centered (F)
Triclinic	$a_1 \neq a_2 \neq a_3$ $\alpha_{12} \neq \alpha_{23} \neq \alpha_{31}$				
Monoclinic	$a_1 \neq a_2 \neq a_3$ $\alpha_{23} = \alpha_{31} = 90°$ $\alpha_{12} \neq 90°$				
Orthorhombic	$a_1 \neq a_2 \neq a_3$ $\alpha_{12} = \alpha_{23} = \alpha_{31} = 90°$				
Tetragonal	$a_1 = a_2 \neq a_3$ $\alpha_{12} = \alpha_{23} = \alpha_{31} = 90°$				
Trigonal	$a_1 = a_2 = a_3$ $\alpha_{12} = \alpha_{23} = \alpha_{31} < 120°$				
Cubic	$a_1 = a_2 = a_3$ $\alpha_{12} = \alpha_{23} = \alpha_{31} = 90°$				
Hexagonal	$a_1 = a_2 \neq a_3$ $\alpha_{12} = 120°$ $\alpha_{23} = \alpha_{31} = 90°$				

Table 1.1: Bravais lattices in three-dimensions.

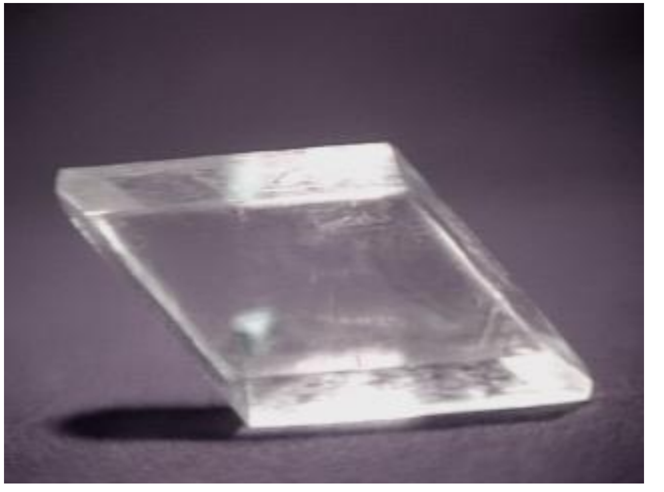
Crystal system	Point group		#	Space groups (international short symbol)
	Hermann-Mauguin	Schönflies		
Triclinic (2)	1	C_1	1	P1
	$\bar{1}$	C_i	2	$P\bar{1}$
Monoclinic (13)	2	C_2	3-5	P2, $P2_1$, C2
	m	C_s	6-9	Pm, Pc, Cm, Cc
	2/m	C_{2h}	10-15	P2/m, $P2_1/m$, C2/m, P2/c, $P2_1/c$, C2/c
Orthorhombic (59)	222	D_2	16-24	P222, $P222_1$, $P2_12_12$, $P2_12_12_1$, $C222_1$, C222, F222, I222, $I2_12_12_1$
	mm2	C_{2v}	25-46	Pmm2, $Pmc2_1$, Pcc2, Pma2, $Pca2_1$, Pnc2, $Pmn2_1$, Pba2, $Pna2_1$, Pnn2, Cmm2, $Cmc2_1$, Ccc2, Amm2, Aem2, Ama2,Aea2, Fmm2, Fdd2, Imm2, Iba2, Ima2
	mmm	D_{2h}	47-74	Pmmm, Pnnn, Pccm, Pban, Pmma, Pnna, Pmna, Pcca, Pbam, Pccn, Pbcm, Pnnm, Pmmn, Pbcn, Pbca, Pnma, Cmcm, Cmce, Cmmm, Cccm, Cmme, Ccce, Fmmm, Fddd, Immm, Ibam, Ibca, Imma
Tetragonal (68)	4	C_4	75-80	P4, $P4_1$, $P4_2$, $P4_3$, I4, $I4_1$
	$\bar{4}$	S_4	81-82	$P\bar{4}$, $I\bar{4}$
	$\bar{4}/m$	C_{4h}	83-88	P4/m, $P4_2/m$, P4/n, $P4_2/n$, I4/m, $I4_1/a$
	422	D_4	89-98	P422, $P42_12$, $P4_122$, $P4_12_12$, $P4_222$, $P4_22_12$, $P4_322$, $P4_32_12$, I422, $I4_122$
	4mm	C_{4v}	99-110	P4mm, P4bm, $P4_2cm$, $P4_2nm$, P4cc, P4nc, $P4_2mc$, $P4_2bc$, I4mm, I4cm, $I4_1md$, $I4_1cd$
	$\bar{4}2m$	D_{2d}	111-122	$P\bar{4}2m$, $P\bar{4}2c$, $P\bar{4}2_1m$, $P\bar{4}2_1c$, $P\bar{4}m2$, $P\bar{4}c2$, $P\bar{4}b2$, $P\bar{4}n2$, $I\bar{4}m2$, $I\bar{4}c2$, $I\bar{4}2m$, $I\bar{4}2d$
	4/mmm	D_{4h}	123-142	P4/mmm, P4/mcc, P4/nbm, P4/nnc, P4/mbm, P4/mnc, P4/nmm, P4/ncc, $P4_2/mmc$, $P4_2/mcm$, $P4_2/nbc$, $P4_2/nnm$, $P4_2/mbc$, $P4_2/mnm$, $P4_2/nmc$, $P4_2/ncm$, I4/mmm, I4/mcm, $I4_1/amd$, $I4_1/acd$
Trigonal (25)	3	C_3	143-146	P3, $P3_1$, $P3_2$, R3
	$\bar{3}$	S_6	147-148	$P\bar{3}$, $R\bar{3}$
	32	D_3	149-155	P312, P321, $P3_112$, $P3_121$, $P3_212$, $P3_221$, R32
	3m	C_{3v}	156-161	P3m1, P31m, P3c1, P31c, R3m, R3c
	$\bar{3}m$	D_{3d}	162-167	$P\bar{3}1m$, $P\bar{3}1c$, $P\bar{3}m1$, $P\bar{3}c1$,$R\bar{3}m$, $R\bar{3}c$,
Hexagonal (27)	6	C_6	168-173	P6, $P6_1$, $P6_5$, $P6_2$, $P6_4$, $P6_3$
	$\bar{6}$	C_{3h}	174	$P\bar{6}$
	6/m	C_{6h}	175-176	P6/m, $P6_3/m$
	622	D_6	177-182	P622, $P6_122$, $P6_522$, $P6_222$, $P6_422$, $P6_322$
	6mm	C_{6v}	183-186	P6mm, P6cc, $P6_3cm$, $P6_3mc$
	$\bar{6}m2$	D_{3h}	187-190	$P\bar{6}m2$, $P\bar{6}c2$, $P\bar{6}2m$, $P\bar{6}2c$
	6/mmm	D_{6h}	191-194	P6/mmm, P6/mcc, $P6_3/mcm$, $P6_3/mmc$
Cubic (36)	23	T	195-199	P23, F23, I23, $P2_13$, $I2_13$
	$m\bar{3}$	T_h	200-206	$Pm\bar{3}$, $Pn\bar{3}$, $Fm\bar{3}$, $Fd\bar{3}$, $Im\bar{3}$, $Pa\bar{3}$, $Ia\bar{3}$
	432	O	207-214	P432, $P4_232$, F432, $F4_132$, I432, $P4_332$, $P4_132$, $I4_132$
	$\bar{4}3m$	T_d	215-220	$P\bar{4}3m$, $F\bar{4}3m$, $I\bar{4}3m$, $P\bar{4}3n$, $F\bar{4}3c$, $I\bar{4}3d$
	$m\bar{3}m$	O_h	221-230	$Pm\bar{3}m$, $Pn\bar{3}n$, $Pm\bar{3}n$, $Pn\bar{3}m$, $Fm\bar{3}m$, $Fm\bar{3}c$, $Fd\bar{3}m$, $Fd\bar{3}c$, $Im\bar{3}m$, $Ia\bar{3}d$

Crystals - examples

Gypsum:

Crystal system monoclinic

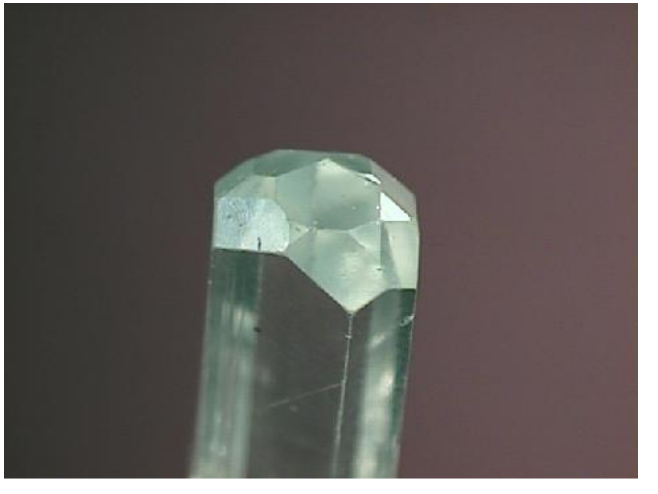
Space group $C_{2h}^{6} = C2/c$



Aquamarine:

Crystal system hexagonal

Space group $D_{6h}^{4} = P6_3/mmc$



Helical symmetry

Examples:

Screw, DNA – double helix

Infinite helical symmetry

2-fold helical symmetry

Helical symmetry

Rotation & translation

- Infinite helical symmetry
- *n*-fold helical symmetry
- Non-repeating helical symmetry

Symmetries

Thank you for your attention